Math 971 Algebraic Topology

April 26, 2005

The Hurewicz map $H : \pi_1(X) \to H_1(X)$ induces, when X is path-connected, an isomorphism from $\pi_1(X)/[\pi_1(X), \pi_1(X)]$ to $H_1(X)$. This result can be used in two ways; knowing a (presentation for) $\pi_1(X)$ allows us to compute $H_1(X)$, by writing the relators additively, giving $H_1(X)$ as the free abelian group on the generators, modulo the kernel of the "presentation matrix" given by the resulting linear equations. Conversely, knowing $H_1(X)$ provides information about $\pi_1(X)$. For example, a calculation on the way to invariance of domain implied that for every knot K in S^3 (i.e., the image of an embedding $h : S^1 \hookrightarrow S^3$), $H_1(S^3 \setminus K) \cong bbz$. This implies that the abelianization of $G_K = \pi_1(S^3 \setminus K)$ (i.e., the largest abelian quotient of G_K is $\mathbb{Z}$. But this in turn implies that for every integer $n \geq 2$, there is a unique surjective homomorphism $G_K \to \mathbb{Z}_n$, since such a homomorphism must factor through the abelianization, and there is exactly one surjective homomorphism $\mathbb{Z} \to \mathbb{Z}_n$! Consequently, there is a unique (normal) subroup (the kernel of this homomorphism) $K_n \subseteq G_K$ with quotient $\mathbb{Z}_n$. Using the Galois correspondence, there is a (unique) covering space X_n of $X = S^3 \setminus K$ corresponding to K_n, called the n-fold cyclic covering of K . This space is determined by K and n, and so its homology groups are determined by the same data. And even though homology cannot distinguish between two knot complements, K, K', it might be the case that homology can distinguish between their cyclic coverings. Consequently, if $H_1(X_n) \not\cong H_1(X'_n)$, then K and K' have non-homeomorphic complement, and so represent "different" embeddings, hence different knots. In practice, one can compute presentations for $\pi_1(X_n)$ (in several different ways), and so one can compute $H_1(X_n)$, providing an effective way to use homology to distinguish knots! This approach was ultimately formalized (by Alexander) into a polynomial invariant of knots, known as the Alexander polynomial.

Computing the homology of the cyclic coverings can be done in several ways. The Reidemeister-Schreier method will allow one to compute a presentation for the kernel of a homomorphism $\varphi : G \to H$, given a presentation of G and a *transversal* of the map, which is a representative of each coset of G modulo the kernel. Abelianizing this will give homology computation. Another approach uses *Seifert surfaces*, orientable surfaces with $\partial\Sigma = K$, to cut $S^3 \setminus K$ open along. Writing $S^3 \setminus K = (S^3 \setminus N(\Sigma)) \cup N(\Sigma)$ allows us to use Mayer-Vietoris to compute homology. But the cyclic covering spaces can be built by "unwinding" this view of $S^3 \setminus K$; instead of gluing the two ends of $N(K)$ to the same $S^3 \setminus N(\Sigma)$, we can take n copies of $S^3 \setminus N(\Sigma)$ and glue them together in a circle. Mayer-Vietoris again tells us how to compute the homology of the resulting space. Details may be found on the accompanying pages taken from Rolfsen's "Knots and Links".