**Math 971 Algebraic Topology**

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The isomorphism between simplicial and singular homology provides very quick proofs of several results about singular homology, which would other would require some effort:

*If the $\Delta$-complex $X$ has no simplices in dimension greater than $n$, then $H_i(X) = 0$ for all $i > n$.*

This is because the simplicial chain groups $C_i^{\Delta}(X)$ are 0, so $H_i^{\Delta}(X) = 0$ .

*If for each $n$, the $\Delta$-complex $X$ has finitely many $n$-simplices, then $H_n(X)$ is finitely generated for every $n$.*

This is because the simplicial chain groups $C_n^{\Delta}(X)$ are all finitely generated, so $H_n^{\Delta}(X)$, being a quotient of a subgroup, is also finitely generated. [We are using here that the number of generators of a subgroup $H$ of an *abelian* group $G$ is no larger than that for $G$; this is not true for groups in general!]

Some more topological results with homological proofs: The Klein bottle and real projective plane cannot embed in $\mathbb{R}^3$. This is because a surface $\Sigma$ embedded in $\mathbb{R}^3$ has a (the proper word is *normal*) neighborhood $N(\Sigma)$, which deformation retracts to $\Sigma$; literally, it is all points within a (uniformly) short distance in the normal direction from the point on the surface $\Sigma$. Our non-embeddedness result follows (by contradiction) from applying Mayer-Vietoris to the pair $(A, B) = (\overline{N(\Sigma)}, \overline{\mathbb{R}^3 \setminus N(\Sigma)})$, whose intersection is the boundary $F = \partial N(\Sigma)$ of the normal neighborhood. The point, though, is that $F$ is an orientable surface; the outward normal (pointing away from $N(\Sigma)$) at every point, taken as the first vector of a right-handed orientation of $\mathbb{R}^3$ allows us to use the other two vectors as an orientation of the surface. So $F$ is one of the surface $F_g$ above whose homologies we just computed. This gives the LES $\widetilde{H}_2(\mathbb{R}^3) \to \widetilde{H}_1(F) \to \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \to \widetilde{H}_1(\mathbb{R}^3)$ which renders as $0 \to \mathbb{Z}^{2g} \to \widetilde{H}(\Sigma) \oplus G \to 0$ , i.e., $\mathbb{Z}^{2g} \cong \widetilde{H}(\Sigma) \oplus G$ . But for the Klein bottle and projective plane (or any closed, non-orientable surface for that matter), $\widetilde{H}_1(\Sigma)$ has torsion, so it cannot be the direct summand of a torsion-free group! So no such embedding exists. This result holds more generally for any 2-complex $K$ whose (it turns out it would have to be first) homology has torsion; any embedding into $\mathbb{R}^3$ would have a neighborhood deformation retracting to $K$, with boundary a (for the exact same reasons as above) closed orientable surface.

**Invariance of Domain:** If $\mathcal{U} \subseteq \mathbb{R}^n$ and $f : \mathcal{U} \to \mathbb{R}^n$ is continuous and injective, then $f(\mathcal{U}) \subseteq \mathbb{R}^n$ is open.

We will approach this through the **Brouwer-Jordan Separation Theorem:** an embedded $(n-1)$-sphere in $\mathbb{R}^n$ separates $\mathbb{R}^n$ into two path components. And for this we need to do a slightly unusual homology calculation:

For $k < n$ and $h : I^k \to S^n$ an embedding of a $k$-cube in to the $n$-sphere, $\widetilde{H}_i(S^n \setminus h(I^k)) = 0$ for all $i$.

Here $I = [-1, 1]$ . The proof proceeds by induction on $k$. For $k = 0$, $S^n \setminus h(I^k) \cong \mathbb{R}^n$, and the result follows. Now suppose the result os true for all embeddings of $C = I^{k-1}$, but is false for some embedding $h : I^k \to S^n$ and some $i$. Then if we divide the cube along its last coordinate, say, as $I^{k-1} \times [-1, 0] = C \times [-1, 0]$ and $C \times [0, 1]$, we can set $A = S^n \setminus h(C \times [-1, 0])$, $B = S^n \setminus h(C \times [0, 1])$, $A \cup B = S^n \setminus h(C \times \{0\})$, and $A \cap B = S^n \setminus h(I^k)$ . These sets are all open, since the image under $h$ of the various sets is compact, hence closed. By hypothesis, $A \cup B = S^n \setminus h(C \times \{0\})$ has trivial reduced homology, while $A \cap B = S^n \setminus h(I^k)$ has non-trivial reduced homology in some dimension $i$. Then the Mayer-Vietoris sequence

$$\cdots \to \widetilde{H}_{i+1}(A \cup B) \to \widetilde{H}_i(A \cap B) \to \widetilde{H}_i(A) \oplus \widetilde{H}_i(B) \to \widetilde{H}_i(A \cup B) \to \cdots$$

reads $0 \to \widetilde{H}_i(A \cap B) \to \widetilde{H}_i(A) \oplus \widetilde{H}_i(B) \to 0$ so $\widetilde{H}_i(A \cap B) \cong \widetilde{H}_i(A) \oplus \widetilde{H}_i(B)$ , so at least one of the groups on the right must be non-trivial, as well. WOLOG $\widetilde{H}_i(B) = \widetilde{H}(S^n \setminus h(C \times [0, 1])) \neq 0$. Even more, choosing (once and for all) a non-zero element $[z] \in \widetilde{H}_I(A \cap B)$, snce its image in the direct sum is non-zero, it's coordinate in (say) $\widetilde{H}_i(B)$ is non-zero.