Math 971 Algebraic Topology

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We have so far introduced two homologies; simplicial, H_*^Δ, whose computation "only" required some linear algebra, and singular, H_*, which is formally less difficult to work with, and which, you may suspect by now, is also becoming less difficult to compute... For Δ-complexes, these homology groups are the same, $H_n^\Delta(X) \cong H_n(X)$ for every X. In fact, the isomorphism is induced by the inclusion $C_n^\Delta(X) \subseteq C_n(X)$. And we have now assembled all of the tools necessary to prove this. Or almost; we need to note that most of the edifice we have built for singular homology could have been built for simplicial homology, including relative homology (for a sub-Δ-complex A of X), and a SES of chain groups, giving a LES sequence for the pair,

$\cdots \to H_n^\Delta(A) \to H_n^\Delta(X) \to H_n^\Delta(X,A) \to H_{n-1}^\Delta(A) \to \cdots$

The proof of the isomorphism between the two homologies proceeds by first showing that the inclusion induces an isomorphism on k-skeleta, $H_n^\Delta(X^{(k)}) \cong H_n(X^{(k)})$, and this goes by induction on k using the Five Lemma applied to the diagram

$$\begin{array}{ccccccccc}
H_{n+1}^\Delta(X^{(k)},X^{(k-1)}) & \to & H_n^\Delta(X^{(k-1)}) & \to & H_n^\Delta(X^{(k)}) & \to & H_n^\Delta(X^{(k)},X^{(k-1)}) & \to & H_{n-1}^\Delta(X^{(k-1)}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{n+1}(X^{(k)},X^{(k-1)}) & \to & H_n(X^{(k-1)}) & \to & H_n(X^{(k)}) & \to & H_n(X^{(k)},X^{(k-1)}) & \to & H_{n-1}(X^{(k-1)})
\end{array}$$

The second and fifth vertical arrows are, by an inductive hypothesis, isomorphisms. The first and fourth vertical arrows are isomorphisms because, essentially, we can, in each case, identify these groups. $H_n(X^{(k)},X^{(k-1)}) \cong H_n(X^{(k)}/X^{(k-1)}) \cong \widetilde{H}_n(\vee S^k)$ are either 0 (for $n \neq k$) or $\oplus\mathbb{Z}$ (for $n = k$), one summand for each n-simplex in X. But the same is true for $H_n^\Delta(X^{(k)},X^{(k-1)})$; and for $n = k$ the generators are precisely the n-simplices of X. The inclusion-induced map takes generators to generators, so is an isomorphism. So by the Five Lemma, the middle rows are also isomorphisms, completing our inductive proof.

Returning to $H_n^\Delta(X) \xrightarrow{I_*} H_n(X)$, we wish now to show that this map is an isomorphism. Any $[z] \in H_n(X)$ is represented by a cycle $z = \sum a_i\sigma_i$ for $\sigma_i : \Delta^n \to X$. But each $\sigma_i(\Delta^n)$ is a compact subset of X, and so meets only finitely-many cells of X. This is true for every singular simplex, and so there is a k for which all of the simplices map into $X^{(k)}$, and so we may treat $z \in C_n(X^{(k)}$. Thought of in this way, it is still a cycle, and so $[z] \in H_n(X^{(k)}) \cong H_n^\Delta(X^{(k)})$ so there is a $z' in C_n^\Delta(X^{(k)})$ and a $w \in C_{n+1}(X^{(k)})$ with $i_\# z' - z = \partial w$. But thinking of $z' in C_n^\Delta(X)$ and $w \in C_{n+1}(X)$, we have the same equality, so $[z'] \in H_n^\Delta(X)$ and $i_*[z'] = [z]$. So i_* is surjective. If $i_*([z]) = 0$, then the cycle $z = \sum a_i\sigma_i$ is a sum of characteristic maps of n-simplices of X, and so can be thought of as an element of $C_n^\Delta(X^n))$. Being 0 in $H_n(X)$, $z = \partial w$ for some $w \in C_{n+1}(X)$. But as before, $w \in C_n(X^r))$ for some r, and so thought of as an element of the image of the isomorphism $i_* : H_n^\Delta(X^{(r)}) \to H_n(X^{(r)})$, $i_*([z]) = 0$, so $[z] = 0$. So $z = \partial u$ for some $u \in C_{n+1}^\Delta(X^r)) \subseteq C_{n+1}^\Delta(X)$. So $[z] = 0$ in $H_n^\Delta(X)$. Consequently, simplicial and singular homology groups are isomorphic.

One consequence of this fact is that we can prove the topological invariance of the **Euler characteristic** of a space X. If X is a Δ-complex made up of a finite number of simplices, then we can count the number m_i of i-simplices in the Δ-complex structure of X. The Euler characteristic of X is then defined to be the alternating sum $\chi(X) = \displaystyle\sum_{i=0}^{\infty}(-1)^i m_i$. Now, as a topological space, X can be given many different Δ-complex structures, and $\chi(X)$ is a priori a number which depends on the structure, not just on X. But once we note that m_i = the rank of the (simplicial) chain group $C_i^\Delta(X)$ (there is one generator for each i-simplex), we find that $\chi(X) = \sum_{i=0}^N(-1)^i\ \mathrm{rank}(C_i(X))$, and then the following result from homological algebra establishes the topological invariance of this number:

Proposition: If $\cdots 0 \to C_n \to \cdots \to C_1 \to C_0 \to 0$ is a chain complex, with every chain group having finite rank, then

$\sum_{i=0}^n(-1)^i\ \mathrm{rank}(C_i) = \sum_{i=0}^n(-1)^i\ \mathrm{rank}(H_i(\mathcal{C})$.

The proof follows from cleverly applying the fact that since $H_i(\mathcal{C})$ =ker∂_i/im∂_{i+1}, z_i = rank(ker∂_i) = rank($H_i(\mathcal{C})$) + rank(im∂_{i+1}) = $h_i + b_{i+1}$, so $h_i = z_i - b_{i+1}$, together with the fact that since (by Noether) $\mathrm{im}(\partial_i) \cong C_i/\mathrm{ker}(\partial_i)$, so $c_i = \mathrm{rank}(C_i) = z_i + b_i$. We therefore have

$\sum_{i=0}^{n}(-1)^i\ \mathrm{rank}(H_i(\mathcal{C}) = \sum_{i=0}^{n}(-1)^i h_i = \sum_{i=0}^{n}(-1)^i(z_i - b_{i+1}) = \sum_{i=0}^{n}(-1)^i z_i - \sum_{i=0}^{n}(-1)^i b_{i+1} = \sum_{i=0}^{n}(-1)^i z_i + \sum_{i=0}^{n}(-1)^i b_i = \sum_{i=0}^{n}(-1)^i(z_i + b_i) = \sum_{i=0}^{n}(-1)^i\ \mathrm{rank}(C_i)$ as desired.

Consequently, $\chi(X) = \sum_{i=0}^{N}(-1)^i\ \mathrm{rank}(C_i^{\Delta}(X)) = \sum_{i=0}^{N}(-1)^i\ \mathrm{rank}(H_i^{\Delta}(X)) = \sum_{i=0}^{N}(-1)^i\ \mathrm{rank}(H_i(X))$, which is an invariant of X, since the singular homology groups are!

The fact that this number has two different interpretations leads to some non-trivial results. First, it tells us that the Euler charactistic calculation is independent of how we express a space X as a Δ-complex. χ is also actually invariant under homotopy equivalence, since the homology groups are; so homotopy equivalent spaces have the same Euler *chi*. Consequently, all contractible spaces, for example, must have Euler characteristic = 1.

Next, by the lifting criterion, if $p : \widetilde{X} \to X$ is a k-fold covering space of a Δ-complex X, then $\widetilde{X}$ can be given a Δ-complex structure with k times as many i-simplices as X, for every i (lift the characteristic maps of the simplices of X....). So $\chi(\widetilde{X}) = k \cdot \chi(X)$. This give a necessary condition for one space to be a covering of another; it's Euler χ must be a multiple of the other. For example, from our homology calculations, it follows that for a closed orientable surface F_g of genus g, $\chi(F_g) = 2 - 2g$. So a k-fold covering of F_g will have Euler χ equal to $k(2-2g) = 2k - 2kg = 2 - 2(kg - k + 1)$, and so is a surface of genus $kg - k + 1$. [The converse, that a surface with this genus k-fold covers F_g, can be established by building the coverings directly.] Consequently, F_5 is a 2-fold covering of F_3, so there is a subgroup of index 2 of $\pi_1(F_3)$ isomorphic to $\pi_1(F_5)$, but F_6 is not a finite-sheeted cover of F_3, because $-4 \not| -10$. [It is also not an inifinite-sheeted covering, because their total spaces are non-compact...] Consequently, $\pi_1(F_6)$ is not isomorphic to a subgroup of $\pi_1(F_3)$.