Math 971 Algebraic Topology
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Homology on "small" chains = singular homology: If $\{\mathcal{U}_\alpha\}$ is an open cover of X, then the inclusions $i_n : C_n^{\mathcal{U}}(X) \to C_n(X)$ induce isomorphisms on homology. This leads to the **Mayer-Vietoris Sequence**: If $X = \mathcal{U} \cup \mathcal{V}$ is the union of two open sets, then the short exact sequences $0 \to C_n(\mathcal{U}\cap\mathcal{V}) \to C_n(\mathcal{U}) \oplus C_n(\mathcal{V}) \to C_n^{\{\mathcal{U},\mathcal{V}\}}(X) \to 0$, together with the isomorphism above, give the long exact sequence

$$\cdots \to H_n(\mathcal{U}\cap\mathcal{V}) \stackrel{(i_{\mathcal{U}*},-i_{\mathcal{V}*})}{\to} H_n(\mathcal{U}) \oplus H_n(\mathcal{V}) \stackrel{j_{\mathcal{U}*}+j_{\mathcal{V}*}}{\to} H_n(X) \stackrel{\partial}{\to} H_{n-1}(\mathcal{U}\cap\mathcal{V}) \to \cdots$$

And just like Seifert - van Kampen, we can replace open sets by sets A, B having neigborhoods which deformation retract to them, and whose intersection deformation retracts to $A \cap B$. For example, subcomplexes $A, B \subseteq X$ of a CW-complex, with $A \cup B = X$ have homology satisfying a long exact sequence $\cdots \to H_n(A\cap B) \stackrel{(i_{A*},-i_{B*})}{\to} H_n(A) \oplus H_n(B) \stackrel{j_{A*}+j_{B*}}{\to} H_n(X) \stackrel{\partial}{\to} H_{n-1}(A\cap B) \to \cdots$

And this is also true for reduced homology; we just augment the chain complexes used above with the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z} \to 0$, where the first non-trivial map is $a \mapsto (a, -a)$ and the second is $(a, b) \mapsto a + b$.

And then we can do some meaningful calculations! An n-sphere S^n is the union $S^n_+ \cup S^n_-$ of its upper and lower hemispheres, each of which is contractible, and have intersection $S^n_+ \cap S^n_- = S^{n-1}_0$ the equatorial $(n-1)$-sphere. So Mayer-Vietoris gives us the exact sequence

$\cdots \to \widetilde{H}_k(S^n_+) \oplus \widetilde{H}_k(S^n_-) \to \widetilde{H}_k(S^n) \to \widetilde{H}_{k-1}(S^{n-1}_0) \to \widetilde{H}_{k-1}(S^n_+) \oplus \widetilde{H}_{k-1}(S^n_-) \to \cdots$, i.e, $0 \to \widetilde{H}_k(S^n) \to \widetilde{H}_{k-1}(S^{n-1}_0) \to 0$ i.e., $\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-1}(S^{n-1})$ for every k and n. So by induction, $\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-n}(S^0) \cong \mathbb{Z}$, if $k = n$ and $= 0$ otherwise.

The 2-torus $T^2 = S^1 \times S^1$ can be thought of as the union of two copies of an annulus $S^1 \times I$, glued together along their (pair of) boundary circles. The resulting LES is $\widetilde{H}_2(S^1\times I) \oplus \widetilde{H}_2(S^1\times I) \to \widetilde{H}_2(T^2) \to \widetilde{H}_1(S^1 \coprod S^1) \to \widetilde{H}_1(S^1\times I) \oplus \widetilde{H}_1(S^1\times I) \to \widetilde{H}_1(T^2) \to \widetilde{H}_0(S^1 \coprod S^1) \to \widetilde{H}_0(S^1\times I) \oplus \widetilde{H}_0(S^1\times I)$ which renders as $0 \to \widetilde{H}_2(T^2) \to \mathbb{Z}\oplus\mathbb{Z} \stackrel{\varphi}{\to} \mathbb{Z}\oplus\mathbb{Z} \to \widetilde{H}_1(T^2) \to \mathbb{Z} \to 0$

We need to know more about the map $\mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}$. The first group has generators consisting of the generators of each of the S^1 path components of $A \cap B$ (represented by the singular 1-simplex wrapping exactly once around the circle), and are each sent to a generator for each of the $S^1 \times I$. Since φ was chosen to be $(i_{A*}, -i_{B*})$, we find that φ has matrix (1,1 ; -1,-1) , which has image spanned by $[1, 1]^T$ and kernel spanned by $[1, 1]^T$. From exactness and a few Noether isomorphism theorems, we can cut up our long exact sequence above as

$$0 \to \widetilde{H}_2(T^2) \to \ker\varphi \to 0 \text{ and } 0 \to (\mathbb{Z}\oplus\mathbb{Z})/im\ \varphi \to \widetilde{H}_1(T^2) \to \mathbb{Z} \to 0$$

(since the first map is onto its image, and the second to last map is injective, once we mod out by its kernel). The first implies that $\widetilde{H}_2(T^2) \cong \mathbb{Z}$, and the second (since our basis for the image extends to a basis for $\mathbb{Z}^2$) becomes $0 \to \mathbb{Z} \to \widetilde{H}_1(T^2) \to \mathbb{Z} \to 0$. This implies that $\widetilde{H}_2(T^2) \cong \mathbb{Z}^2$, because of the

Fact: if $0 \to K \stackrel{\varphi}{\to} G \stackrel{\psi}{\to} H \to 0$ is exact and there is a homomorphism $\rho : H \to G$ with $\psi\rho =$Id , then $G \cong K \times H$. The **proof** consists of defining $\sigma : K \times H \to G$ by $\sigma(k, h) = \varphi(k) + \rho(h)$. As the sum of two homomorphisms it is a homomorphism. If $\sigma(k, h) = \varphi(k) + \rho(h) = 0$ then $0 = \psi\sigma(k, h) = \psi\varphi(k) + \psi\rho(h) = 0 + h = h$, so $0 = \sigma(k, h) = \varphi(k) + \rho(h) = \varphi(k)$, so $k = 0$ by the injectivity of φ. So $(k, h) = (0, 0)$. For surjectivity, given $g \in G$, let $h = \psi(g)$; then $\psi(g - \rho h) = \psi g - \psi\rho h = h - h = 0$, so there is a $k \in K$ with $\varphi k = g - \rho h$, so $\sigma(k, h) = \varphi k + \rho h = g$.

[There are several results like this; the conclusion says that the SES *splits* or is *split exact*; the map ρ provides one sufficient condition for splitness

Consequently, $\widetilde{H}_i(T^2) = \mathbb{Z}$ for $i = 2$, $\mathbb{Z}^2$ for $i = 1$, and 0 for all other i (since T^2 is path-connected, and for $i \geq 3$, our LES reads $\to \widetilde{H}_i(T^2) \to 0$).

The computation for the Klein bottle K^2 is similar; it can be expressed as a pair of annuli $S^1 \times I$ glued along their boundaries, but one of the gluings is by a reflection. The associated inclusion-induced homomorphism, in exactly one case, is $-$Id, not Id; and so the resulting matrix, for one choice of generators, is $(1, 1\ ;\ -1, 1)$. After row and column reducton, this becomes $(1, 0\ ;\ 0, 2)$. This matrix has no kernel, so, using the same cutting up process, $0 \to \widetilde{H}_2(K^2) \to \ker\varphi \to 0$ and $0 \to (\mathbb{Z}\oplus\mathbb{Z})/im\varphi \to \widetilde{H}_1(K^2) \to \mathbb{Z} \to 0$ becomes $0 \to \widetilde{H}_2(K^2) \to 0$ and $0 \to \mathbb{Z}_2 \to \widetilde{H}_1(K^2) \to \mathbb{Z} \to 0$ so $\widetilde{H}_2(K^2) = 0$ and $\widetilde{H}_1(K^2) \cong \mathbb{Z}\oplus\mathbb{Z}_2$. As before, all other (reduced) homology groups are 0.

For the real projective plane P^2, we can express it as a Möbius band M with a disk D glued to its boundary. Their intersection is a circle S^1. Writing the Mayer-Vietoris sequence in this situation gives $0 \to \widetilde{H}_2(P^2) \to \mathbb{Z} \to 0 \oplus \mathbb{Z} \to \widetilde{H}_1(P^2) \to 0$.

Again we need to know more about the middle map $i_* : \widetilde{H}_1(S^1) \to \widetilde{H}_1(M)$ in order to determine the unknown groups. M deformation retracts to its central circle, and the generator for $\widetilde{H}_1(\partial M)$, wrapping once around ∂M, is sent to a map wrapping around twice, and so represents $2\times$generator $\in \widetilde{H}_1(M)$. So the middle map is injective, with image $2\mathbb{Z}$. And so $\widetilde{H}_2(P^2) = 0$, and $\widetilde{H}(P^2) \cong \mathbb{Z}/\text{im}(i_*) \cong \mathbb{Z}_2$. All other groups, as before, are 0.