Math 971 Algebraic Topology

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Perhaps the most important property of the fundamental group is that a continuouos map between spaces induces a homomorphism between groups. (This implied, for instance, that homeomorphic spaces have isomorphic π_1). The same is true for homology groups, for essentially the same reason. Given a map $f : X \to Y$, there is an induced map $f_\# : C_n(X) \to C_n(Y)$ defined by postcomposition; for a singular simplex σ, $f_\#(\sigma) = f \circ \sigma$, and we extend the map linearly. Since $f \circ (g|_A) = (f \circ g)|_A$ (postcomposition commutes with restriction of the domain), $f_\#$ commutes with ∂ : $f_\#(\partial\sigma) = \partial(f_\#(\sigma))$. A homomorphism between chain complexes (i.e., a sequence of such maps, one for each chain group) which commutes with the boundaries maps this way, is called a *chain map*. A chain map, such as $f_\#$, therefore, takes cycles to cycles, and boundaries to boundaries, and so $f_\# : Z_i(X) \to Z_i(Y)$ (which is linear, so a homomorphism) induces a homomorphism $f_* : H_i(X) \to H_i(Y)$ by $f_*[z] = [f_\#(z)]$. Since it is defined by composition with singular simplices, it is immediate that, for a map $g : Y \to Z$, $(g \circ f)_* = g_* \circ f_*$. And since the identity map $I : X \to X$ satisfies $I_\# = Id$, so $I_* = Id$, homeomorphic spaces have isomorphic homology groups.

Another important property of π_1 is that homotopic maps give the same induced map (after correcting for basepoints). This is also true for homology; if $f \sim g : X \to Y$, then $f_* = g_*$. The proof, however, is not quite as straightforward as for homotopy. And it requires some new technology; the chain homotopy. A chain homotopy H between the chain complexes $f_\#, g_\# : C_*(X) \to C_*(Y)$ is a sequence of homomorphisms $H_i : C_i(X) \to C_{i+1}(Y)$ satisfying $H_{i-1}\partial_i + \partial_{i+1}H_i = f_\# - g_\# : C_i(X) \to C_i(Y)$. Having H implies that $f_* = g_*$, since for an i-cycle z (with $\partial_i(z) = 0$) we have $f_*[z] - g_*[z] = [f_\#(z) - g_\#(z)] = [H_{i-1}\partial_i(z) + \partial_{i+1}H_i(z)] = [H_{i-1}(0) + \partial_{i+1}(w)] = [\partial_{i+1}(w)] = 0$. And the existence of a homotopy between f and g implies the existence of a chain homotopy between $f_\#$ and $g_\#$. This is because the homotopy gives a map $H : X \times I \to Y$, which induces a map $H_\# : C_{i+1}(X \times I) \to C_{i+1}(Y)$. Then we pull, from our back pocket, a *prism map* $P : C_i(X) \to C_{i+1}(X \times I)$; the composition $H_\# \circ P$ will be our chain homotopy. The prism map takes a (singular) i-simplex σ and sends it to a sum of singular $(i+1)$-simplices in $X \times I$. and the way we define it is to take the i-simplex Δ^i, and taking it to $\Delta^i \times I$ (i.e., a *prism*), and thinking of this as a sum of $(i+1)$-simplices. Using the map $\sigma' = \sigma \times Id : \Delta^i \times I \to X \times I$ restricted to each of these $(i+1)$-simplices yields the prism map. Now, there are many ways of decomposing a prism into simplices, but we need to be careful to choose one which restricts well to each of the faces of Δ^i, in order to get the chain homotopy property we require. In the end, what this requires is that the decomposition, when restricted to any face of Δ^i (which we think of as a copy of Δ^{i-1}), is the same as the decomposition we would have applied to a prism over an $(i-1)$-simplex. After some exploration, we are led to the following formulation.

If we write $\Delta^n \times \{0\} = [v_0, \dots, v_n]$ and $\Delta^n \times \{1\} = [w_0, \dots, w_n]$, then we can decompose $\Delta^n \times I$ as the (n+1)-simplices $[v_0, \dots, v_i, w_i, \dots, w_n]$. We then define $P(\sigma) = \sum(-1)^i \sigma'|_{[v_0,\dots,v_i,w_i,\dots,w_n]}$. A routine calculation verifies that $(\partial P + P\partial)(\sigma) = \sigma'|_{[w_0,\dots,w_n]} - \sigma'|_{[v_0,\dots v_n]}$; Composing with $H_\#$ yields our result.

Consequently, for example, homotopy equivalent spaces have isomporphic (reduced) homology groups; homotopy equivalences induce isomorphisms. So all contractible spaces have trivial reduced homology in all dimensions, since they are all homotopy to a point. If we think of a cell complex as a collection of disks glued together, this lends some hope that we can compute their homology groups, since we can compute the homology of the building blocks. Our next goal is to make turn this idea into action; but we need another tool, to frame our answer in the best way possible.

Exact sequences: Most of the fundamental properties of homology groups are described in terms of exact sequences. A sequence of homomorphisms $\cdots \overset{f_{n+1}}{\to} A_n \overset{f_n}{\to} A_{n-1} \overset{f_{n-1}}{\to} a_{n-2} \to \cdots$ of abelian groups is called *exact* if $\mathrm{im}(f_n) = \ker(f_{n-1})$ for every n. In most cases, we get the most mileage out of an exact sequence when some of the groups (or more generally, some of the maps) are trivial; $0 \to A \overset{f}{\to} B$ is exact $\Leftrightarrow$ f is injective (and the same if A receives the 0 map), and $A \overset{f}{\to} B \to 0$ is exact $\Leftrightarrow$ f is surjective (and the same if the map with domain B is the 0 map). An exact sequence $0 \to A \to B \to C \to 0$ is called a *short exact sequence*.