Math 971 Algebraic Topology

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Singular homology groups are very quick to define, but what do they measure? The basic idea is that we are trying to mimic simplicial homology, but because a general topological space X cannot be thought of as being built out of simplices, we do the next best thing; we study the space by mapping simplices in. Formally, this is what we did with simplicial homology anyway, except that we restricted ourselves to a very few special singular simplices (the characteristic maps of the building blocks for X). In the end an n-cycle $\sum a_i\sigma_i^n$, since the faces of the σ_i must match up precisely, in order to cancel in the sum, can be thought of as a map of an n-complex into X, made by gluing the n-simplices σ_i together before mapping in. The fact that faces cancel really means that these simplices restrict to the same maps on their faces. The integer coefficients can really be interpreted as taking multiple copies of Δ^n and gluing them together along their boundaries (the signs tell us the underlying orientations). The idea is that this n-complex is being mapped "around a hole", unless it extends to a map of an $(n+1)$-complex into X (having our n-complex as boundary). So singular homology really is trying to detect holes, it is just doing it with maps.....

The "fun" with singular homology groups, though, comes when you try to compute them. $C_n(X) = \{\sum a_i\sigma_i : a_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \to X$ is continuous$\}$ is typically a huge group, since there will be immense numbers of maps $\Delta^n \to X$. About the only space for which this is not true is the one-point space $*$; then there is, for each n, exactly one (distinct) map $\sigma_n : \Delta^n \to *$; the constant map. Therefore each face of Δ^n gives the same restriction map σ^{n-1}, and so the boundary maps can be dirctly computed (the depend on the parity of the number $n+1$ of faces an n-simplex has). We find that $\partial_{2n} = Id$ and $\partial_{2n-1} = 0$. so in computing homology groups, we either have kernel everything ($\partial_i = 0$) and image everything ($\partial_{i+1} = Id$) or kernel nothing ($\partial_i = Id$) and image nothing ($\partial_{i+1} = 0$), so in both cases $H_i(*) = 0$. Except for $i = 0$; then $\partial_0 = 0$ (by definition) and $\partial_1 = 0$, so $H_0(*) = \mathbb{Z}$. But other than this example (and, well, OK, spaces with the discrete topology; it's the same calculation as above for every point!), computing singular homology from the definition is quite a chore! so we need to build some labor-saving devices, namely, some theorems to help us break the problem of computing these groups into smaller, more managable pieces.

First set of managable pieces: if we decompose X into its path components, $X = \bigcup X_\alpha$, then $H_i(X) \cong \bigoplus H_i(X_\alpha)$ for every i. This is because every singular simplex, since Δ^i is path-connected, maps into some X_α . So $C_i(X) \cong \bigoplus C_i(X_\alpha)$. Since the boundary of a simplex mapping into X_α consists of simplices in X_α, the boundary maps respect the decomposistions of the chain groups, so $B_i(X) \cong \bigoplus B_i(X_\alpha)$ and $Z_i(X) \cong \bigoplus Z_i(X_\alpha)$, and so the quotients are $H_i(X) \cong \bigoplus H_i(X_\alpha)$.

So, if we wish to, we can focus on computing homology groups for path-connected spaces X. For such a space, $H_0(X) \cong \mathbb{Z}$, generated by any map of a 0-simplex (= a point) into X. This is because any pair of 0-simplices are homologous; given any two points $x, y \in X$, there is a path $\gamma : I \to X$ from x to y, This path can be interpreted as a singular 1-simplex, and $\partial\gamma = y - x$. So $H_0(X)$ is generated by a single point $[x]$. No multiple of this point is null-homologous, because for any 1-chain $\sum n_i\sigma_i$, the sum of the coefficients of its boundary is 0 (since this is true for each singular 1-simplex), and any 0-chain $\sum n_i[x_i]$ is homologous to $(\sum n_i)[x]$ by the above argument.

A small techincal aside: the fact that $H_0(*) = \mathbb{Z}$ is annoying to some, and often requires treating 0-dimensional homology as a special case. But since the boundary of a singular 1-simplex is always of the form $v - w$, we find that the image of ∂_1 is always contained in the subgroup of $C_0(X)$ consisting of chains whose coefficients sum to 0. This means that we can, for free, *augment* the singular chain complex by a map $\cdots \to C_1(X) \overset{\partial_1}{\to} C_0)X) \overset{\alpha}{\to} \mathbb{Z} \to 0$ where α is the map $\alpha(\sum a_i\sigma_i^0) = \sum a_i$. This is still a chain complex (compositions of consecutive maps are 0); the resulting homology groups are called *reduced* homology $\widetilde{H}_i(X)$. The only affect this really has is to remove one copy of $\mathbb{Z}$ from H_0; $\widetilde{H}_0(X) \oplus \mathbb{Z} \cong H_0(X)$. All other homology groups are unchanged.