

First: A discussion of the relative merits of Δ -complexes versus simplicial complexes. Then:

The *orientation* on a simplex of X : Each simplex σ^n is determined by a map $f : \{v_0, \dots, v_n\} \rightarrow X^{(0)}$; an orientation on σ^n is an (equivalence class of) the ordered $(n+1)$ -tuple $(f(v_0), \dots, f(v_n)) = (V_0, \dots, V_n)$. Another ordering of the same vertices represents the same orientation if there is an *even* permutation taking the entries of the first $(n+1)$ -tuple to the second. This should be thought of as a generalization of the right-hand rule for \mathbb{R}^3 , interpreted as orienting the vertices of a 3-simplex. Note that there are precisely two orientations on a simplex.

Now to define homology! We start by defining *n-chains*; these are (finite) formal linear combinations of the (oriented!) n -simplices of X , where $-\sigma$ is interpreted as σ with the opposite (i.e., other) orientation. Adding formal linear combinations formally, we get the n -th *chain group* $C_n(X) = \{\sum n_\alpha \sigma_\alpha : \sigma_\alpha \text{ an oriented } n\text{-simplex in } X\}$. We next define a *boundary operator* $\partial : C_n(X) \rightarrow C_{n-1}(X)$, whose image will be the $(n-1)$ -chains that are the “boundaries” of n -chains. The idea is that the boundary of a 2-simplex, for example, should be a “sum” of its three faces (since they do make up the boundary of the simplex), but we need to take into account their orientations, in order to be getting the correct sum. Thinking of the orientation on a 1-simplex $[v, w]$ as an arrow pointing from v to w , we are led to believe that the boundary of a 2-simplex $[u, v, w]$ should be $[u, v] + [v, w] + [w, u]$. Similarly, the boundary of $[u, v]$, on reflection, should be $[v] - [u]$, to distinguish the head of the arrow (the $+$ side) from the tail (the $-$ side). On the basis of these examples, trying to find a consistent formula, one might eventually be led to the following formulation. We define the boundary on the basis elements $\sigma_\alpha = \sigma$ of $C_n(X)$ as $\partial\sigma = \sum (-1)^i \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, v_n]}$, where $\sigma : [v_0, \dots, v_n] \rightarrow X$ is the characteristic map of σ_α . $\partial\sigma$ is therefore an alternating sum of the faces of σ . We then extend the definition by linearity to all of $C_n(X)$. When a notation indicating dimension is needed, we write $\partial = \partial_n$. We define $\partial_0 = 0$.

This definition, it turns out, is cooked up to make the maxim “boundaries have no boundary” true; that is, $\delta_{n-1} \circ \delta_n = 0$, the 0 map. This is because, for any simplex $\sigma = [v_0, \dots, v_n]$,

$$\delta \circ \delta(\sigma) = \delta\left(\sum_{i=0}^n (-1)^i \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, v_n]}\right) = \left(\sum_{j < i} (-1)^j (-1)^i \sigma|_{[v_0, \dots, \widehat{v_j}, \dots, \widehat{v_i}, \dots, v_n]}\right) + \left(\sum_{j > i} (-1)^{j-1} (-1)^i \sigma|_{[v_0, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots, v_n]}\right)$$

The distinction between the two pieces is that in the second part, v_j is actually the $(j-1)$ -st vertex of the face. Switching the roles of i and j in the second sum, we find that the two are negatives of one another, so they sum to 0, as desired.

And this little calculation is all that it takes to define homology groups! What this tells us is that $\text{im}(\delta_{n+1}) \subseteq \ker(\delta_n)$ for every n . $\ker(\delta_n) = Z_n(X)$ are called the *n-cycles* of X ; they are the n -chains with 0 (i.e., empty) boundary. They form a (free) abelian subgroup of $C_n(X)$. $\text{im}(\delta_{n+1}) = B_n(X)$ are the *n-boundaries* of X ; they are, of course, the boundaries of $(n+1)$ -chains in X . The n -th homology group of X , $H_n(X)$ is the quotient $Z_n(X)/B_n(X)$; it is an abelian group. This consists, essentially, of the n -cycles that are not boundaries; i.e., they represent the “holes” in X .

A key observation is that the boundary maps δ_n are linear, that is, they are homomorphisms between the free abelian groups $\delta_n : C_n(X) \rightarrow C_{n-1}(X)$. Consequently, they can be expressed as (integer-valued) matrices Δ_n . Row reducing Δ_n (over the integers!) allows us to find a basis v_1, \dots, v_k for $Z_n(X)$ (clearing denominators to get vectors over \mathbb{Z}). Then since $\Delta_n \Delta_{n+1} = 0$, the columns of Δ_{n+1} are in the kernel of Δ_n , so can be expressed as linear combinations of the v_i . These combinations can be determined by row reducing the augmented matrix $(v_1 \cdots v_k | \Delta_{n+1})$. This will row reduce to $\left(\begin{array}{c|c} I & C \\ 0 & 0 \end{array} \right)$, and C basically describes the boundaries $B_n(X)$ in terms of the basis v_1, \dots, v_k .

The homology group $H_n(X)$ is then the *cokernel* of C , i.e., $\mathbb{Z}^k / \text{im} C$. Note that C will have integer entries, since we know that the columns of Δ_{n+1} can be expressed as integer linear combinations of the v_i , and, being a basis, there is only one such expression.