Math 971 Algebraic Topology

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Every space it X the quotient of its universal cover (if it has one!) by its fundamental group $G = \pi_1(X,x_0)$, realized as the group of deck transformations. And the quotient map is the covering projection. So $X \cong \widetilde{X}/G$. In general, a quotient of a space Z by a group action G need not be a covering map; the action must be *properly discontinuous*, which means that for every point $z \in Z$, there is a neighborhood $\mathcal{U}$ of x so that $g \neq 1 \Rightarrow \mathcal{U} \cap g\mathcal{U} = \emptyset$ (the group action carries sufficiently small neighborhoods off of themselves). The evenly covered neighborhoods provide these for the universal cover. And conversely, the quotient of a space by a p.d. group action is a covering space.

But! Given $G = \pi_1(X,x_0)$ and its action on a universal cover $\widetilde{X}$, we can, instead of quotienting out by G, quotient out by any <u>subgroup</u> H of G, to build $X_H = \widetilde{X}/H$. This is a space with fundamental group H, having $\widetilde{X}$ as universal covering. And since the quotient (covering) map $p_G : \widetilde{X} \to X = \widetilde{X}/G$ factors through $\widetilde{X}/H$, we get an induced map $p_H : \widetilde{X}/H \to X$, which is a covering map; open sets with trivial inclusion-induced homomorphism lift homeomorphically to $\widetilde{X}$, hence homeomorphically to $\widetilde{X}/H$; taking lifts to each point inverse of $x \in X$ verifies the evenly covering property for p_H . So every subgroup of G is the fundamental group of a covering of X.

We can further refine this to give the *Galois correspondence.* Two covering spaces $p_1 : X_1 \to X$, $p_2 : X_2 \to X$ are *isomorphic* if there is a homeomorphism $h : X_1 \to X_2$ with $p_1 = p_2 \circ h$. Choosing basepoints x_1, x_2 mapping to $x_0 \in X$, this implies that, if $h(x_1) = x_2$, then $p_{1*}(\pi_1(X_1,x_1)) = p_{2*}(h_*(\pi_1(X_1,x_1))) = p_{2*}(\pi_1(X_2,x_2))$. On the other hand, our homeomorphism h need not take our chosen basepoints to one another; if $h(x_1) = x_3$, then $p_{1*}(\pi_1(X_1,x_1)) = p_{2*}(\pi_1(X_2,x_3))$. But $p_{2*}(\pi_1(X_2,x_2))$ and $p_{2*}(\pi_1(X_2,x_3))$ are isomorphic, via a change of basepoint isomorphism $\widehat{\eta}$, where η is a path in X_2 from x_2 to x_3. But such a path projects to X has a loop at x_0, and since the change of basepoint isomorphism is by "conjugating" by the path η, the resulting groups $p_{2*}(\pi_1(X_2,x_2))$ and $p_{2*}(\pi_1(X_2,x_3))$ are conjugate, by $p_2 \circ \eta$. So, without reference to basepoints, isomorphic coverings give, under projection, conjugate subgroups of $\pi_1(X,x_0)$. But conversely, given covering spaces X_1, X_2 whose subgroups $p_{1*}(\pi_1(X_1,x_1))$ and $p_{2*}(\pi_1(X_2,x_2))$ are conjugate, lifting a loop γ representing the conjugating element to a loop $\widetilde{\gamma}$ in X_2 starting at x_2 gives, as its terminal endpoint, a point x_3 with $p_{1*}(\pi_1(X_1,x_1)) = p_{2*}(\pi_1(X_2,x_3))$ (since it conjugates back!), and so, by the lifting criterion, there is an isomorphism $h : (X_1,x_1) \to (X_2,x_3)$. So conjugate subgroups give isomorphic coverings. Thus, for a path-connected, locally path-connected, semi-locally simply-connected space X, the image of the induced homomorphism on π_1 gives a one-to-one correspondence between [isomorphism classes of (connected) coverings of X] and [conjugacy classes of subgroups of $\pi_1(X)$].

So, for example, if you have a group G that you are interested in, you know of a (nice enough) space X with $\pi_1(X) \cong G$, and you know enough about the covering of X, then you can gain information about the subgroup structure of G. For example, and in some respects as motivation for all of this machinery!, a free group $F(\Sigma)$ is π_1 of a bouquet of circles X. Any covering space $\widetilde{X}$ of X is a union of vertices and edges, so is a graph. Collapsing a maximal tree to a point, $\widetilde{X}$ is $\simeq$ a bouquet of circles, so has free π_1. So, every subgroup of a free group is free. (That is a lot shorter than the original, group-theoretic, proof...)