Math 971 Algebraic Topology

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The number of sheets of a covering map can also be determined from the fundamental groups:

Proposition: If X and $\widetilde{X}$ are path-connected, then the number of sheets of a covering map equals the index of the subgroup $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)$ in $G = \pi_1(X, x_0)$.

To see this, choose loops $\{\gamma_\alpha\}$ representing representatives $\{g_\alpha\}$ of each of the (right) cosets of H in G. Then if we lift each of them to loops based at $\widetilde{x}_0$, they will have distinct endpoints; if $\widetilde{\gamma}_1(1) = \widetilde{\gamma}_2(1)$, then by uniqueness of lifts, $\gamma_1 * \overline{\gamma_2}$ lifts to $\widetilde{\gamma}_1 * \overline{\widetilde{\gamma}_2}$, so it lifts to a loop, so $\gamma_1 * \overline{\gamma_2}$ represents an element of $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)$, so $g_1 = g_2$. Conversely, every point in $p^{-1}(x_0)$ is the endpoint of on of these lifts, since we can construct a path $\widetilde{\gamma}$ from $\widetilde{x}_0$ to any such point y, giving a loop $\gamma = p \circ \widetilde{\gamma}$ representing an element $g \in \pi_1(X, x_0)$. But then $g = hg_\alpha$ for some $h \in H$ and α, so γ is homotopic rel endpoints to $\eta * \gamma_\alpha$ for some loop η representing h. But then lifting these based at $\widetilde{x}_0$, by hmotopy lifting, $\widetilde{\gamma}$ is homotopic rel endpoints to $\widetilde{\eta}$, which is a loop, followed by the lift $\widetilde{\gamma}_\alpha$ of γ_α starting at $\widetilde{x}_0$. So $\widetilde{\gamma}$ and $\widetilde{\gamma}_\alpha$ have the same value at 1.

Therefore, lifts of representatives of coset representatives of H in G give a 1-to-1 correspondence, given by $\widetilde{\gamma}(1)$, with $p^{-1}x_0$. In particular, they have the same cardinality.

The path lifting property (because $\pi([0,1], 0) = \{1\}$) is actually a special case of a more general **lifting criterion**: If $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering map, and $f : (Y, y_0) \to (X, x_0)$ is a map, where Y is path-connected and locally path-connected, then there is a lift $\widetilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ of f (i.e., $f = p \circ \widetilde{f}$) $\Leftrightarrow f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. Furthermore, two lifts of f which agree at a single point are equal.

If the lift exists, then $f = p \circ \widetilde{f}$ implies $f_* = p_* \circ \widetilde{f}_*$, so $f_*(\pi_1(Y, y_0)) = p_*(\widetilde{f}_*(\pi_1(Y, y_0))) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$, as desired. Conversely, if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$, then we wish to build the lift of f. Not wishing to waste our work on the special case, we will *use* path lifting to do it! Given $y \in Y$, choose a path γ in Y from y_0 to y and use path lifting in X to lift the path $f \circ \gamma$ to a path $\widetilde{f \circ \gamma}$ with $\widetilde{f \circ \gamma}(0) = \widetilde{x}_0$. Then define $\widetilde{f}(y) = \widetilde{f \circ \gamma}(1)$. Provided we show that this is well-defined and continuous, it is our required lift, since $(p \circ \widetilde{f})(y) = p(\widetilde{f}(y)) = p(\widetilde{f \circ \gamma}(1)) = p \circ \widetilde{f \circ \gamma})(1) = (f \circ \gamma)(1) = f(\gamma(1)) = f(y)$. To show that it is well-defined, if η is any other path from y_0 to y, then $\gamma * \overline{\eta}$ is a loop in Y, so $f \circ (\gamma * \overline{\eta}) = (f \circ \gamma) * \overline{(f \circ \eta)}$ is a loop in X representing an element of $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$, and so lifts to a loop in $\widetilde{X}$ based at $\widetilde{x}_0$. Consequently, as before, $f \circ \gamma$ and $f \circ \eta$ lift to paths starting at $\widetilde{x}_0$ with the same value at 1. So $\widetilde{f}$ is well-defined. To show that $\widetilde{f}$ is continuous, we use the evenly covered property of p. Given $y \in Y$, and a neighborhood $\widetilde{\mathcal{U}}$ of $\widetilde{f}(y)$ in $\widetilde{X}$, we wish to find a nbhd $\mathcal{V}$ of y with $\widetilde{f}(\mathcal{V}) \subseteq \widetilde{\mathcal{U}}$. Choosing an evenly covered neighborhood $\mathcal{U}_y$ for $f(y)$, choose the sheet $\widetilde{\mathcal{U}}_y$ over $\mathcal{U}_y$ which contains $\widetilde{f}(y)$, and set $\mathcal{W} = \widetilde{\mathcal{U}} \cap \widetilde{\mathcal{U}}_y$. This is open in $\widetilde{X}$, and p is a homeomorphism from this set to the open set $p(\mathcal{W}) \subseteq X$. Then if we set $\mathcal{V}' = f^{-1}(p(\mathcal{W})$ this is an open set containing y, and so contains a path-connected open set $\mathcal{V}$ containing y. Then is for every point $z \in \mathcal{V}$ we build a path γ from y_0 to z by concatenating a path from y_0 to y with a path *in* $\mathcal{V}$ from y to z, then by unique path lifting, since $f(\mathcal{V} \subseteq \mathcal{U}_y$, $f \circ \gamma$ lifts to the concatenation of a path from $\widetilde{x}_0$ to $\widetilde{f}(y)$ and a path *in* $\widetilde{\mathcal{U}}_y$ from $\widetilde{f}(y)$ to $\widetilde{f}(z)$. So $\widetilde{f}(z) \in \widetilde{\mathcal{U}}$.

Because $\widetilde{f}$ is built by lifting paths, and path lifting is unique, the last statement of the proposition follows.

Universal covering spaces: As we shall see, a particularly important covering space to identify is one which is simply connected. One thing we can see from the lifting crierion is that such a covering is essentially unique:

If X is locally path-connected, and has two connected, simply connected coverings $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$, then choosing basepoints $x_i, i = 0, 1, 2$, since $p_{1*}(\pi_1(X_1, x_1)) = p_{2*}(\pi_1(X_2, x_2)) = \{1\} \subseteq \pi_1(X, x_0)$, the lifting criterion with each projection playing the role of f in turn gives us maps $\widetilde{p}_1 : (X_1, x_1) \to (X_2, x_2)$ and $\widetilde{p}_2 : (X_2, x_2) \to (X_1, x_1)$ with $p_2 \circ \widetilde{p}_1 = p_1$ and $p_1 \circ \widetilde{p}_2 = p_2$. Consequently,

$p_2 \circ \widetilde{p}_1 \circ \widetilde{p}_2 = p_1 \circ \widetilde{p}_2 = p_2$ and similarly, $p_1 \circ \widetilde{p}_2 \circ \widetilde{p}_1 = p_2 \circ \widetilde{p}_1 = p_1$. So $\widetilde{p}_1 \circ \widetilde{p}_2 : (X_2, x_2) \rightarrow (X_2, x_2)$, for example, is a lift of p_2 to the covering map p_2. But so is the identity map! By uniqueness, therefore, $\widetilde{p}_1 \circ \widetilde{p}_2 = Id$. Similarly, $\widetilde{p}_2 \circ \widetilde{p}_1 = Id$. So (X_1, x_1) and (X_2, x_2) are homeomorphic. So up to homeomorphism, a space can have only one connected, simply-connected covering space. It is known as the *universal covering* of the space X.

Not every (locall path-connected) space X has a universal covering; a (further) necessary condition is that X be *semi-locally simply connected*. The idea is that If $p : \widetilde{X} \rightarrow X$ is the universal cover, then for every point $x \in X$, we have an evenly-covered neighborhood $\mathcal{U}$ of x. The inclusion $i : \mathcal{U} \rightarrow X$, by definition, lifts to $\widetilde{X}$, so $i_*(\pi_1(\mathcal{U}, x)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}) = \{1\}$, so i_* is the trivial map. Consequently, every loop in $\mathcal{U}$ is null-homotopic in X. This is semi-local simple connectivity; every point has a neighborhood whose inclusion-induced homomorphism is trivial. Not all spaces have this property; the most famous is the Hawaiian earrings $X = \bigcup_n \{x \in \mathbb{R}^2 : ||x - (1/n, 0)|| = 1/n\}$. The point $(0, 0)$ has no such neighborhood.