Math 971 Algebraic Topology

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Gluing groups: given groups G_1, G_2, with subgroups H_1, H_2 that are isomorphic $H_1 \cong H_2$, how can we "glue" G_1 and G_2 together along their "common" subgroup? More generally (and with our eye on van Kampen's Theorem) given a group H and homomorphisms $\phi_1 : H \to G_i$, we wish to build the largest group "generated" by G_1 and G_2, in which $\phi_1(h) = \phi_2(h)$ for all $h \in H$.

Idea: start with $G_1 * G_2$ (to get the first part), and then take a quotient to insure that $\phi_1(h)(\phi_2(h))^{-1} = 1$ for every h. Using presentations $G_1 =< \Sigma_1 | R_1 >$, $G_2 =< \Sigma_2 | R_2 >$, we can do this as

$$G = (G_1 * G_2)/ < \phi_1(h)(\phi_2(h))^{-1} : h \in H >^N =< \Sigma_1 \coprod \Sigma_2 | R_1 \cup R_2 \cup \{\phi_1(h)(\phi_2(h))^{-1} : h \in H\} >$$

This group $G == G_1 *_H G_2$ is the *largest* group generated by G_1 and G_2 in which $\phi_1(h) = \phi_2(h)$ for all $h \in H$, and is called the *amalgamated free product* or *free product with amalgamation (over H)* . [**Warning!** Group theorists will generally use this term only if both homoms ϕ_1, ϕ_2 are injective. Some people use the term *pushout* in our more general setting.]

Important special cases : $G *_H \{1\} = G/ < \phi(H) >^N =< \Sigma | R \cup \phi(H) >$, and $G_1 *_{\{1\}} G_2 \cong G_1 * G_2$

The relevance to π_1: the **Seifert-van Kampen Theorem**. If we express a topological space as the union $X = X_1 \cup X_2$, then we have inclusion-induced homomorphisms $j_{1*} : \pi_1(X_1) \to \pi_1(X)$, $j_{2*} : \pi_1(X_2) \to \pi_1(X)$ - to be precise, we should choose a common basepoint in $A = X_1 \cap X_2$. This gives a homomorphism $\phi : \pi(X_1) * \pi_1(X_2) \to \pi_1(X)$. When

X_1, X_2 are open, and $X_1, X_2, X_1 \cap X_2$ are path-connected

we can see that this homom is onto: Given $x_0 \in X_1 \cap X_2$ and a loop $\gamma : (I, \partial I) \to (X, x_0)$, we wish to show that it is homotopic rel endpoints to a product of loops which lie alternately in X_1 and X_2. The idea: cut I into subintervals which alternately map into X_1 and X_2. Their endpoints, therefore, all map into $X_1 \cap X_2$. Setting $y_k = \gamma(I_k \cap I_{k+1})$, we can, since $X_1 \cap X_2$ is path-connected, find a path $\delta_k : I \to X_1 \cap X_2$ with $\delta_k(0) = y_k$ and $\delta_k(1) = x_0$. Defining $\eta_k = \gamma|_{I_k}$, we have that, in $\pi_1(X, x_0)$,

$$[\gamma] = [\eta_1 * \cdots * \eta_m] = [\eta_1 * (\delta_1 * \overline{\delta_1}) * \eta_2 * \cdots * \eta_{m-1} * (\delta_{m-1} * \overline{\delta_{m-1}}) * \eta_m] = [\eta_1 * \delta_1][\overline{\delta_1} * \eta_2 * \delta_2] \cdots [\overline{\delta_{m-2}} * \eta_{m-1} * \delta_{m-1}][\overline{\delta_{m-1}} * \eta_m]$$

We can insert the $\delta_k * \overline{\delta_k}$ into these products because each is homotopic to the constant map, and η_k*(constant) is homotopic to η_k by the same sort of homotopy that established that the constant map represents the identity in the fundamantal group.

This results in a product of loops (based at x_0) which alternately lie in X_1 and X_2. This product can therefore be interpreted as lying in $\pi(X_1) * \pi_1(X_2)$, and maps, under ϕ, to $[\gamma]$. ϕ is therefore onto, and $\pi_1(X)$ is isomorphic to the free product modulo the kernel of this map ϕ.

Loops $\gamma : (I, \partial I) \to (A, x_0)$, can, using the maps $i_{1*} : \pi_1(A) \to \pi_1(X_1)$, $i_{2*} : \pi_1(A) \to \pi_1(X_2)$, be thought as either in $\pi_1(X_1)$ or $\pi_1(X_2)$. So the word $i_{1*}(\gamma)(i_{2*}(\gamma))^{-1}$, in $\pi(X_1) * \pi_1(X_2)$, is sent to the identity in $\pi_1(X)$ under ϕ. So these elements lie in the kernel of ϕ.

Seifert - van Kampen Theorem: $\ker(\phi) =< i_{1*}(\gamma)(i_{2*}(\gamma))^{-1} : \gamma \in \pi_1(A) >^N$, so $\pi_1(X) \cong \pi_1(X_1) *_{\pi_1(A)} \pi_1(X_2)$.

Before we explore the proof of this, let's see what we can do with it!

Fundamental groups of graphs: Every finite connected graph Γ has a *maximal tree* T, a connected subgraph with no simple circuits. Since any tree is the union of smaller trees joined at a vertex, we can, by induction, show that $\pi_1(T) = \{1\}$. In fact, if e is an outermost edge of T, then T deformation retracts to $T \setminus e$, so, by induction, T is contractible. Consequently (*Hatcher, Proposition 0.17*), Γ and the quotient space Γ/T are homotopy equivalent, and so have the same π_1. But $\Gamma/T = \Gamma_n$ is a bouquet of n circles for some n. If we let $\mathcal{U}$ = a neighborhood of the vertex in Γ_n, which is contractible, then, by singling out one petal of the bouquet, we have $\Gamma_n = (\Gamma_{n-1} \cup \mathcal{U}) \cup (\Gamma_1 \cup \mathcal{U}) = X_1 \cup X_2$ with $\Gamma_k \cup \mathcal{U} \simeq (\Gamma_k \cup \mathcal{U})/\mathcal{U} \cong \Gamma_k$. And since $X_1 \cap X_2 = \mathcal{U} \simeq *$, we have that $\pi_1(\Gamma_n) \cong \pi_1(\Gamma_{n-1}) *_1 \pi_1(\Gamma_1) = \pi_1(\Gamma_{n-1}) * \mathbb{Z}$ So, by induction, $\pi_1(\Gamma) \cong \pi_1(\Gamma_n) \cong \mathbb{Z} * \cdots * \mathbb{Z} = F(n)$, the free group on n letters, where n = the number of edges not in a maximal tree for Γ. The generators for the group consist of the edges not in the tree, prepended and appended by paths to the basepoint.