Math 971 Algebraic Topology

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Gluing groups: given groups G_1, G_2, with subgroups H_1, H_2 that are isomorphic $H_1 \cong H_2$, how can we "glue" G_1 and G_2 together along their "common" subgroup? More generally (and with our eye on van Kampen's Theorem) given a group H and homomorphisms $\phi_1 : H \to G_i$, we wish to build the largest group "generated" by G_1 and G_2, in which $\phi_1(h) = \phi_2(h)$ for all $h \in H$.

We can do this by starting with $G_1 * G_2$ (to get the first part), and then take a quotient to insure that $\phi_1(h)(\phi_2(h))^{-1} = 1$ for every h. Using presentations $G_1 =< \Sigma_1 | R_1 >$, $G_2 =< \Sigma_2 | R_2 >$, if we insist on quotienting out by as little as possible to get our desired result, we can do this very succinctly as

$$G = (G_1 * G_2)/ < \phi_1(h)(\phi_2(h))^{-1} : h \in H >^N =< \Sigma_1 \coprod \Sigma_2 | R_1 \cup R_2 \cup \{\phi_1(h)(\phi_2(h))^{-1} : h \in H\} >$$

This group $G == G_1 *_H G_2$ is the *largest* group generated by G_1 and G_2 in which $\phi_1(h) = \phi_2(h)$ for all $h \in H$, and is called the *amalgamated free product* or *free product with amalgamation (over H)* . [**Warning!** Group theorists will generally use this term only if both homoms ϕ_1, ϕ_2 are injective. (This insures that the natural maps of G_1, G_2 into $G_1 *_H G_2$ are injective.) But we will use this term for all ϕ_1, ϕ_2. (Some people use the term *pushout* in this more general case.)]

Important special cases : $G *_H \{1\} = G/ < \phi(H) >^N =< \Sigma | R \cup \phi(H) >$, and $G_1 *_\{ 1\} G_2 \cong G_1 * G_2$

The relevance to π_1: the Seifert-van Kampen Theorem.

If we express a topological space as the union $X = X_1 \cup X_2$, then we have inclusion-induced homomorphisms

$$j_{1*} : \pi_1(X_1) \to \pi_1(X) \text{ , } j_{2*} : \pi_1(X_2) \to \pi_1(X)$$

- to be precise, we should choose a common basepoint in $A = X_1 \cap X_2$. This in turn gives a homomorphism $\phi : \pi(X_1) * \pi_1(X_2) \to \pi_1(X)$. Under the hypotheses

$$X_1, X_2 \text{ are open, and } X_1, X_2, X_1 \cap X_2 \text{ are path-connected}$$

we can see that this homom is onto:

Given $x_0 \in X_1 \cap X_2$ and a loop $\gamma : (I, \partial I) \to (X, x_0)$, we wish to show that it is homotopic rel endpoints to a product of loops which lie alternately in X_1 and X_2. But $\{\gamma^{-1}(X_1), \gamma^{-1}(X_2)\}$ is an open cover of the compact metric space I, and so there is an $\epsilon > 0$ (a *Lebesgue number*) so that every interval of length ϵ in I lies in one of these two sets, i.e., maps, under γ, into either X_1 or X_2. If we set $N = \lceil 1/\epsilon \rceil$, then setting $a_i = i/N$, then we get a sequence of intervals $J_i = [a_i, a_{i+1}], i = 0, \dots N-1$, each mapping into X_1 or X_2. If J_i and J_{i+1} both map into the same subpace, replace them in the sequence with their union. Continuing in this fashion, reducing the number of subintervals by one each time, we will eventually find a collection I_k, $k = 1, \dots m$, of intervals covering I, overlapping only on their endpoints, which alternately map into X_1 and X_2. Their endpoints, therefore, all map into $X_1 \cap X_2$. Setting $y_k = \gamma(I_k \cap I_{k+1})$, we can, since $X_1 \cap X_2$ is path-connected, find a path $\delta_k : I \to X_1 \cap X_2$ with $\delta_k(0) = y_k$ and $\delta_k(1) = x_0$. Choosing our favorite homeomorphisms $h_k : I \to I_k$ and defining $\eta_k = \gamma|_{I_k} \circ h_k$, we have that, in $\pi_1(X, x_0)$,

$$[\gamma] = [\eta_1 * \dots * \eta_m] = [\eta_1 * (\delta_1 * \overline{\delta_1}) * \eta_2 * \dots * \eta_{m-1} * (\delta_{m-1} * \overline{\delta_{m-1}}) * \eta_m]$$
$$= [\eta_1 * \delta_1][\overline{\delta_1} * \eta_2 * \delta_2] \cdots [\overline{\delta_{m-2}} * \eta_{m-1} * \delta_{m-1}][\overline{\delta_{m-1}} * \eta_m]$$

We can insert the $\delta_k * \overline{\delta_k}$ into these products because each is homotopic to the constant map, and $\eta_k *$(constant) is homotopic to η_k by the same sort of homotopy that established that the constant map represents the identity in the fundamantal group.

This results in a product of loops (based at x_0) which alternately lie in X_1 and X_2. This product can therefore be interpreted as lying in $\pi(X_1) * \pi_1(X_2)$, and maps, under ϕ, to $[\gamma]$. ϕ is therefore onto, and $\pi_1(X)$ is isomorphic to the free product modulo the kernel of this map ϕ.

Loops $\gamma : (I, \partial I) \to (A, x_0)$, can, using the inclusion-induced maps $i_{1*} : \pi_1(A) \to \pi_1(X_1)$, $i_{2*} : \pi_1(A) \to \pi_1(X_2)$, be thought as either in $\pi_1(X_1)$ or $\pi_1(X_2)$. So the word $i_{1*}(\gamma)(i_{2*}(\gamma))^{-1}$, in $\pi(X_1) * \pi_1(X_2)$, is set to the identity in $\pi_1(X)$ under ϕ. So these elements lie in the kernel of ϕ.

Seifert - van Kampen Theorem: $\ker(\phi) =< i_{1*}(\gamma)(i_{2*}(\gamma))^{-1} : \gamma \in \pi_1(A) >^N$, so $\pi_1(X) \cong \pi_1(X_1) *_{\pi_1(A)} \pi_1(X_2)$.

Before we explore the proof of this, let's see what we can do with it!

Fundamental groups of graphs: Every finite connected graph Γ has a *maximal tree* T, a connected subgraph with no simple circuits. Since any tree is the union of smaller trees joined at a vertex, we can, by induction, show that $\pi_1(T) = \{1\}$. In fact, if e is an outermost edge of T, then T deformation retracts to $T \setminus e$, so, by induction, T is contractible. Consequently (*Hatcher, Proposition 0.17*), Γ and the quotient space Γ/T are homotopy equivalent, and so have the same π_1. But $\Gamma/T = \Gamma_n$ is a bouquet of n circles for some n. If we let $\mathcal{U}$ = a neighborhood of the vertex in Γ_n, which is contractible, then, by singling out one petal of the bouquet, we have

$$\Gamma_n = (\Gamma_{n-1} \cup \mathcal{U}) \cup (\Gamma_1 \cup \mathcal{U}) = X_1 \cup X_2$$

with $\Gamma_k \cup \mathcal{U} \simeq (\Gamma_k \cup \mathcal{U})/\mathcal{U} \cong \Gamma_k$. And since $X_1 \cap X_2 = \mathcal{U} \simeq *$, we have that

$$\pi_1(\Gamma_n) \cong \pi_1(\Gamma_{n-1}) *_1 \pi_1(\Gamma_1) = \pi_1(\Gamma_{n-1}) * \mathbb{Z}$$

So, by induction, $\pi_1(\Gamma) \cong \pi_1(\Gamma_n) \cong \mathbb{Z} * \cdots * \mathbb{Z} = F(n)$, the free group on n letters, where $n =$ the number of edges not in a maximal tree for Γ. The generators for the group consist of the edges not in the tree, prepended and appended by paths to the basepoint.