Math 971 Algebraic Topology

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Group theory "done right": presentations

Σ = a set; a *reduced word* on Σ is a (formal) product $a_1^{\epsilon_1}\cdots a_n^{\epsilon_n}$ with $a_i\in\Sigma$ and $\epsilon_i=\pm1$, and either $a_i\neq a_{i+1}$ or $\epsilon_i\neq\epsilon_{i+1}$ for every i. (I.e., no $aa^{-1},a^{-1}a$ in the product.)

The free group $F(\Sigma)$ = the set of reduced words, with multiplication = concatenation followed by reduction; remove all possible $aa^{-1},a^{-1}a$ from the site of concatenation.

identity element = the empty word, $(a_1^{\epsilon_1}\cdots a_n^{\epsilon_n})^{-1}=a_n^{-\epsilon_n}\cdots a_1^{-\epsilon_1}$. $F(\Sigma)$ is generated by Σ, with no relations among the generators other than the "obvious" ones.

Important property of free groups: any function $f:\Sigma\to G$, G a group, extends uniquely to a homomorphism $\phi:F(\Sigma)\to G$.

If $R\subseteq F(\Sigma)$, then $<R>^N$ = normal subgroup generated by R

$= \{\prod_{i=1}^n g_ir_ig_i^{-1} : n\in\mathbb{N}_0, g_i\in F(\Sigma), r_i\in R\}$

=smallest normal subgroup containing R.

$F(\Sigma)/<R>^N$ = the group with *presentation* $<\Sigma|R>$; it is the largest quotient of $F(\Sigma)$ in which the elements of R are the identity. Every group has a presentation:

$$G=F(G)/<gh(gh)^{-1}:g,h\in G>^N$$

where (gh) is interpreted as a single letter in G.

If $G_1=<\Sigma_1|R_1>$ and $G_2=<\Sigma_2|R_2>$, then their *free product* $G_1*G_2=<\Sigma_1\coprod\Sigma_2|R_1\cup R_2>$ (Σ_1,Σ_2 must be treated as (formally) disjoint). Each element has a unique reduced form as $g_1\cdots g_n$ where the g_i alternate from G_1,G_2. G_1,G_2 can be thought of as subgroups for G_1*G_2, in the obivous way. Important property of free products: any pair of homoms $\phi_i:G_i\to G$ extends uniquely to a homom $\phi:G_1*G_2\to G$ (exactly the way you think it does).