**Math 971 Algebraic Topology**

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## Group theory "done right": presentations

$\Sigma$ = a set; a *reduced word* on $\Sigma$ is a (formal) product $a_1^{\epsilon_1}\cdots a_n^{\epsilon_n}$ with $a_i \in \Sigma$ and $\epsilon_i = \pm 1$, and either $a_i \neq a_{i+1}$ or $\epsilon_i \neq \epsilon_{i+1}$ for every $i$. (I.e., no $aa^{-1}, a^{-1}a$ in the product.)

The free group $F(\Sigma)$ = the set of reduced words, with multiplication = concatenation followed by reduction; remove all possible $aa^{-1}, a^{-1}a$ from the site of concatenation.

identity element = the empty word, $(a_1^{\epsilon_1}\cdots a_n^{\epsilon_n})^{-1} = a_n^{-\epsilon_n}\cdots a_1^{-\epsilon_1}$. $F(\Sigma)$ is generated by $\Sigma$, with no relations among the generators other than the "obvious" ones.

Important property of free groups: any function $f : \Sigma \to G$ , $G$ a group, extends uniquely to a homomorphism $\phi : F(\Sigma) \to G$.

If $R \subseteq F(\Sigma)$, then $< R >^N$ = normal subgroup generated by $R$

$= \{\prod_{i=1}^{n} g_i r_i g_i^{-1} : n \in \mathbb{N}_0, g_i \in F(\Sigma), r_i \in R\}$

=smallest normal subgroup containing $R$.

$F(\Sigma)/ < R >^N$ = the group with *presentation* $< \Sigma | R >$ ; it is the largest quotient of $F(\Sigma)$ in which the elements of $R$ are the identity. Every group has a presentation:

$$G = F(G)/ < gh(gh)^{-1} : g, h \in G >^N$$

where $(gh)$ is interpreted as a single letter in $G$.

If $G_1 =< \Sigma_1 | R_1 >$ and $G_2 =< \Sigma_2 | R_2 >$, then their *free product* $G_1 * G_2 =< \Sigma_1 \coprod \Sigma_2 | R_1 \cup R_2 >$ ($\Sigma_1, \Sigma_2$ must be treated as (formally) disjoint). Each element has a unique reduced form as $g_1 \cdots g_n$ where the $g_i$ alternate from $G_1, G_2$. $G_1, G_2$ can be thought of as subgroups for $G_1 * G_2$, in the obivous way. Important property of free products: any pair of homoms $\phi_i : G_i \to G$ extends uniquely to a homom $\phi : G_1 * G_2 \to G$ (exactly the way you think it does).