Math 971 Algebraic Topology

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Group theory "done right": presentations

Σ = a set; a *reduced word* on Σ is a (formal) product $a_1^{\epsilon_1} \cdots a_n^{\epsilon_n}$ with $a_i \in \Sigma$ and $\epsilon_i = \pm 1$, and either $a_i \neq a_{i+1}$ or $\epsilon_i \neq \epsilon_{i+1}$ for every i. (I.e., no $aa^{-1}, a^{-1}a$ in the product.)

The free group $F(\Sigma)$ = the set of reduced words, with multiplication = concatenation followed by reduction; remove all possible $aa^{-1}, a^{-1}a$ from the site of concatenation.

identity element = the empty word, $(a_1^{\epsilon_1} \cdots a_n^{\epsilon_n})^{-1} = a_n^{-\epsilon_n} \cdots a_1^{-\epsilon_1}$. $F(\Sigma)$ is generated by Σ, with no relations among the generators other than the "obvious" ones.

Important property of free groups: any function $f : \Sigma \to G$, G a group, extends uniquely to a homomorphism $\phi : F(\Sigma) \to G$.

If $R \subseteq F(\Sigma)$, then $< R >^N$ = normal subgroup generated by R
$= \{\prod_{i=1}^{n} g_i r_i g_i^{-1} : n \in \mathbb{N}_0, g_i \in F(\Sigma), r_i \in R\}$
=smallest normal subgroup containing R.

$F(\Sigma)/ < R >^N$ = the group with *presentation* $< \Sigma | R >$; it is the largest quotient of $F(\Sigma)$ in which the elements of R are the identity. Every group has a presentation:

$$G = F(G)/ < gh(gh)^{-1} : g, h \in G >^N$$

where (gh) is interpreted as a single letter in G.

If $G_1 =< \Sigma_1 | R_1 >$ and $G_2 =< \Sigma_2 | R_2 >$, then their *free product* $G_1 * G_2 =< \Sigma_1 \coprod \Sigma_2 | R_1 \cup R_2 >$ (Σ_1, Σ_2 must be treated as (formally) disjoint). Each element has a unique reduced form as $g_1 \cdots g_n$ where the g_i alternate from G_1, G_2. G_1, G_2 can be thought of as subgroups for $G_1 * G_2$, in the obivous way. Important property of free products: any pair of homoms $\phi_i : G_i \to G$ extends uniquely to a homom $\phi : G_1 * G_2 \to G$ (exactly the way you think it does).