Math 872 Problem Set 2

Starred (*) problems are due Thursday, February 5.

9. [Hatcher p.40, #20] If $H : X \times I \to X$ is a homotopy with $H_0 = H_1 =$ the identity map, show that $\gamma(t) = H(x_0, t)$ is a loop representing an element $g = [\gamma] \in \pi_1(X, x_0)$ lying in the center of $\pi_1(X, x_0)$., i.e., $gh = hg$ for all $h \in \pi_1(X, x_0)$

(*) 10. Show that $\langle a, b \mid a^5 = b^4, aba = bab \rangle$ is a(nother) presentation for the trivial group (that is, show that the relations imply that $a = 1$ and $b = 1$).

11. (a) Show that the group $\langle a \mid a^n \rangle$ is isomorphic to the cyclic group $\mathbb{Z}_n$ on n elements.

    (b) Show that the group $G = \langle a, b, c \mid a^2, b^2, c^2, (ab)^3, (ac)^2, (bc)^3 \rangle$ is isomorphic to the symmetric group S_4 on four letters. [You can/should model this on our demonstration for S_3; find a surjective homomorphism to S_4, and show that G has at most 24 elements!]

(*) 12. [Hatcher, p.52, #1] Show that the free product $G * H$ of non-trivial groups has trivial center; if $x \in G * H$ and $xy = yx$ for every $y \in G * H$, then $x = 1$.

    [Hint: show that if $x \neq 1$ then there is an element that it <u>doesn't</u> commute with! Think of group elements as products alternately in G and H.]

13. Show that if $G = \langle A \mid R \rangle$, $x \notin A$, and w is a word in the alphabet A, then $H = \langle A \cup \{x\} \mid R \cup \{xw\} \rangle$ is isomorphic to G. That is, build homomorphisms $G \to H$ and $H \to G$ that are inverses of one another.

    [This is one of the two *Tietze transformations*, that describe how to change presentations without changing the group.]

14. Show that $X = \mathbb{R}^2 \setminus \mathbb{Q}^2 \subseteq \mathbb{R}^2$ is path connected, and $\pi_1(X)$ is uncountable. (I.e., find uncountably many loops no two of which are homotopic to one another.)

(*) 15. Compute (i.e., find a presentation for) the fundamental group of the complement X of the coordinate axes in $\mathbb{R}^3$, i.e., $X = \{(x, y, z) \in \mathbb{R}^3 \ : \ |xy| + |xz| + |yz| \neq 0\}$.

    [Hint: Find a 'simpler' space, homotopy equivalent to X, to compute the fundamental group of.]

16. Show that if $f : \partial D^n \to X$ is the attaching map of an n-cell D^n, with $n \geq 3$, then the inclusion $X \hookrightarrow X \cup_f D^n$ induces an isomorphism on π_1. Show that the same is true if we attach any (finite or infinite) collection of $n \geq 3$ cells.