Euler characteristics: Given a finite Δ-complex X (that is, having a finite number of simplices), the number of simplices in each dimension (which is also the rank of each of the groups in the simplicial chain complex) are not an invariant of the space, but, it turns out, the alternating sum $\chi(X) = \sum(-1)^i \dim C_i^\Delta(X)$ is a topological invariant, called the *Euler characteristic* of X. This also is a consequence of the isomrphism between singular and simplicial homology (since it implies that the simplicial homology groups are topological invariants), together with the following result:

Proposition: If $0 \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0$ is a chain complex of abelian groups, with each chain group having finite rank, and having homology groups $H_i = H_i(\mathcal{C})$, then $\sum(-1)^i \mathrm{r}ank(C_i) = \sum(-1)^i \mathrm{r}ank(H_i)$.

The proof consists of noting that if we set $Z_i = \ker \partial_i \subseteq C_i$ and $B_i = \mathrm{i}m\partial_{i+1} \subseteq Z_i$, then by one of the Noether isomorphism theorems $B_{i-1} \cong C_i/Z_i$, while $H_i = Z_i/B_i$ by definition, so

$\mathrm{r}ank(B_{i-1}) = \mathrm{r}ank(C_i) - \mathrm{r}ank(Z_i)$, so $\mathrm{r}ank(C_i) = \mathrm{r}ank(Z_i) + \mathrm{r}ank(B_{i-1})$,
while $\mathrm{r}ank(H_i) = \mathrm{r}ank(Z_i) - \mathrm{r}ank(B_i)$, so

$\sum(-1)^i \mathrm{r}ank(C_i) - \sum(-1)^i \mathrm{r}ank(H_i)$
$= \sum(-1)^i[(\mathrm{r}ank(Z_i) + \mathrm{r}ank(B_{i-1})) - (\mathrm{r}ank(Z_i) - \mathrm{r}ank(B_i))]$
$= \sum(-1)^i[\mathrm{r}ank(B_{i-1}) + \mathrm{r}ank(B_i)]$
$= \sum(-1)^i[\mathrm{r}ank(B_i) - (-1)^{i-1}\mathrm{r}ank(B_{i-1})]$
$= \sum(-1)^i \mathrm{r}ank(B_i) - \sum(-1)^i \mathrm{r}ank(B_i) = 0$, as desired.

Therefore $\chi(X) = \sum(-1)^i \mathrm{r}ank(H_i^\Delta(X)) = \sum(-1)^i \mathrm{r}ank(H_i(X))$ depends only on X, not on a particular Δ-complex structure.

As a special case, since $\chi(D^n) = 1$ (since the only non-trivial homnology group of D^n is $H_0(D^n) = \mathbb{Z}$), we obtain Euler's formula: if a 2-disk is triangulated with v vertices, e edges, and f faces, then $v - e + f = 1$. Since the singular homology groups are invariant under homotopy equivalence, we also have that every contractible finite Δ-complex has Euler characteristic 1. So a quick (but only partially successful) way to show that a connected Δ-complex X is not contractible is to show that $\chi(X) \neq 1$.

We can also extend these results to finite CW-complexes; the alternating sum of the number of cells in each dimension is an invariant of the underlying topological space. To show this, we introduce yet "another" homology theory, *celluluar homology*. The interesting feature of this is that the chain groups are singular homology groups! Specifically, if X is a CW-complex with k-skeleta $X^{(k)}$, then the relative singular homology group $H_k(X^{(k)}, X^{(k-1)}) \cong \widetilde{H}_k(X^{(k)}/X^{(k-1)}) \cong \widetilde{H}_k(\vee S^k) \cong \oplus\mathbb{Z}$, with one $\mathbb{Z}$ summand for each k-cell of X. From the exact sequence of the triple $(X^{(k)}, X^{(k-1)}, X^{(k-2)})$ we have a connecting homomorphism $d_k : H_k(X^{(k)}, X^{(k-1)}) \to H_{k-1}(X^{(k-1)}, X^{(k-2)})$, which as with the sequence for a pair, "really" takes a relative cycle $[z]$, under

$$C_k(X^{(k)}, X^{(k-1)}) \xrightarrow{\partial} C_{k-1}(X^{(k-1)}) \xrightarrow{p} C_{k-1}(X^{(k-1)}, X^{(k-2)})$$

to the coset of $[\partial z]$. Applying this twice,

$$H_k(X^{(k)}, X^{(k-1)}) \to H_{k-1}(X^{(k-1)}, X^{(k-2)}) \to H_{k-2}(X^{(k-2)}, X^{(k-3)}),$$

is therefore zero, since it amounts to taking the ordinary boundary twice. So we have a chain complex $\{C_n^{CW}(X), d_n\} = \{H_n(X^{(n)}, X^{(n-1)}), d_n\}$, called the *cellular chain complex* of X, whose homology groups are the *cellular homology groups of X*.

The "Euler characteristic" of the cellular complex is the alternating sum of the number of n-cells in X, which by the homological argument above is the same as the alternating sum of the ranks of the cellular homology groups. As with simplicial homology, cellular homology is defined in terms of a particular CW-structure on X, but again, we can show that it is independent of this structure, by showing that cellular homology is isomorphic to singular homology. To do this, we first need some basic facts:

(a) $H_n(X^{(n+1)}) \cong H_n(X)$, from the long exact sequence of the pair,

$$0 = H_{n+1}(X, X^{(n+1)}) \to H_n(X^{(n+1)}) \to H_n(X) \to H_n(X, X^{(n+1)}) = 0,$$

since $H_i(X, X^{(n+1)}) \cong \widetilde{H}_i(X/X^{(n+1)})$ and

(b) If Y is a connected CW-complex with $Y^{(n)} =$ a point, then $\widetilde{H}_i(Y) = 0$ for $i \leq n$. For finite dimensional Y ($Y = Y^{(k)}$ for some k), this follows by induction, using the long exact sequence for the pair $(Y^{(j+1)}, Y^{(j)})$, since

$$H_{i+1}(Y^{(j+1)}, Y^{(j)}) \to \widetilde{H}_i(Y^{(j)}) \to \widetilde{H}_i(Y^{(j+1)}) \to H_i(Y^{(j+1)}, Y^{(j)})$$

is $0 \to \widetilde{H}_i(Y^{(j)}) \to \widetilde{H}_i(Y^{(j+1)}) \to 0$ for $i \neq j, j+1$, so $\widetilde{H}_i(Y^{(j)}) \cong \widetilde{H}_i(Y^{(j+1)})$ for $i < j$ and $i > j+1$, and $\widetilde{H}_j(Y^{(j)}) \to \widetilde{H}_j(Y^{(j+1)})$ is surjective. On the other hand, $\widetilde{H}_i(Y^{(n)}) = \widetilde{H}(\{*\}) = 0$, so for $i \leq n$ $\quad \widetilde{H}_i(Y) = \widetilde{H}_i(Y^{(k)}) \cong \cdots \cong \widetilde{H}_i(Y^{(n+1)}) \twoheadleftarrow \widetilde{H}_i(Y^{(n)}) = 0,\quad$ so $\widetilde{H}_i(Y) = 0$. The above argument also shows that for any CW-complex X, $\widetilde{H}_i(X^{(n)}) = 0$ for $i > n$, since $0 = \widetilde{H}_i(X^{(0)}) \cong \cdots \cong \widetilde{H}_i(X^{(n-1)}) \cong \widetilde{H}_i(X^{(n)})$.

For the infinite dimensional case, we recycle an old argument to show that since any cycle $[z]$ in $\widetilde{H}_i(Y)$ is a finite union of singular simplices, and a compact set meets only finitely many cells, we can think of z as a chain in some $Y^{(k)}$, where by the above it is a boundary, so it is a boundary in Y, so $[z] = 0$, so $\widetilde{H}_i(Y) = 0$.

The same argument above <u>also</u> shows that for $i < n$, $\widetilde{H}_i(X^{(n)}) \cong \widetilde{H}_i(X)$ (under the inclusion-induced homomorphism); for finite-dimensional complexes this requires only $\widetilde{H}_i(X) = \widetilde{H}_i(X^{(k)}) \cong \cdots \cong \widetilde{H}_i(X^{(n+1)}) \cong \widetilde{H}_i(X^{(n)})$, while for infinite-dimensional complexes the same final argument shows injectivity, and a parallel argument [any representative of $[z] \in \widetilde{H}_i(X)$ is really a chain in some $X^{(r)}$, so is in the image of $\widetilde{H}_i(X^{(n)}) \xrightarrow{\cong} \widetilde{H}_i(X^{(r)}) \to \widetilde{H}_i(X)$] proves surjectivity.

With these facts in hand, we proceed to prove that $H_n^{CW}(X) \cong H_n(X)$. The basic idea is that $H_n^{CW}(X)$ is computed from

$H_{n+1}(X^{n+1}, X^n) \overset{d_{n+1}}{\to} H_n(X^n, X^{n-1}) \overset{d_n}{\to} H_{n-1}(X^{n-1}, X^{n-2})$, which is really

$$H_{n+1}(X^{n+1}, X^n) \overset{\partial_{n+1}}{\to} \widetilde{H}_n(X^n) \overset{p_*}{\to} H_n(X^n, X^{n-1}) \overset{\partial_n}{\to} \widetilde{H}_{n-1}(X^{n-1}) \overset{p_*}{\to} H_{n-1}(X^{n-1}, X^{n-2})$$

built from three different LESs of pairs! (This is, however, <u>not</u> exact.)

$\widetilde{H}_n(X) \cong \widetilde{H}_n(X^{(n+1)})$, but $\widetilde{H}_n(X^{(n+1)})$ is part of a long exact sequence

$$H_{n+1}(X^{(n+1)}, X^{(n)}) \overset{\partial_{n+1}}{\to} \widetilde{H}_n(X^{(n)}) \overset{\iota_*}{\to} \widetilde{H}_n(X^{(n+1)}) \to H_n(X^{(n+1)}, X^{(n)})$$

with $H_n(X^{(n+1)}, X^{(n)}) \cong \widetilde{H}_n(X^{(n+1)}/X^{(n)}) = 0$ (its n-skeleton is a point), so $\widetilde{H}_n(X^{(n+1)}) \cong \widetilde{H}_n(X^{(n)})/\ker(\iota_*) = \widetilde{H}_n(X^{(n)})/\mathrm{i}m(\partial_{n+1})$.

The LES of the pair

$$\cdots \to 0 = \widetilde{H}_{n-1}(X^{(n-2)}) \to \widetilde{H}_{n-1}(X^{n-1}) \overset{p_*}{\to} H_{n-1}(X^{n-1}, X^{n-2}) \to \cdots$$

implies that the second map p_* is injective, so $\ker d_n = \ker \partial_n$.

The LES sequence of the pair

$$\cdots \to 0 = \widetilde{H}_n(X^{(n-1)}) \to \widetilde{H}_n(X^n) \overset{p_*}{\to} H_n(X^n, X^{n-1}) \to \cdots$$

implies that the first map p_* is injective, so p_* maps $\mathrm{i}m\ \partial_{n+1}$ isomorphically to $\mathrm{i}m\ d_{n+1} = p_*(\mathrm{i}m\ \partial_{n+1})$, and $\widetilde{H}_n(X^{(n)})$ isomorphically to $\mathrm{i}m\ p_* = \ker \partial_n = \ker d_n$. Consequently, $p_* : \widetilde{H}_n(X^{(n)}) \to \ker d_n$ induces an isomorphism

$$\widetilde{H}_n(X) \cong \widetilde{H}_n(X^{(n+1)}) \cong \widetilde{H}_n(X^{(n)})/\mathrm{i}m\ \partial_{n+1} \xrightarrow{\cong} \ker d_n/\mathrm{i}m\ d_{n+1} = H_n^{CW}(X).$$

As before, this isomorphism immediately leads to some useful facts, both about cellular and singular homology, that are much tougher to establish without the isomorphism:

The cellular homology groups depend only on the underlying topological space, not on the CW structure.

The Euler characteristic of a finite CW-complex is well-defined.

If a CW-complex X has no k-cells, then $H_k(X) = 0$ (since the k-th cellular chain group is 0).

More generally, if a CW-complex X has r k-cells, then $H_k(X)$ has a generating set with at most r elements.

If X is n-dimensional, then $H_n(X)$ is free abelian
(since $H_n^{CW}(X) = \ker d_n \subseteq C_n^{CW}(X)$, since $C_{n+1}^{CW}(X) = 0$, so im $d_{n+1} = 0$).

If a CW-complex X has no $k-2$- and k-cells, then $H_{k-1}(X)$ is the free abelian group on the $k-1$-cells of X (since the chain complex is $0 \to C_{k-1}^{CW}(X) \to 0$ at that point).

Together, the third and sixth facts give a much quicker way to compute the homology groups of spheres S^n (for $n \geq 2$, anyway), for example, since S^n has a CW-structure with one 0-cell and one n-cell. Another quick collection of examples is $S^n \times S^m$ with $n = m \geq 2$ (1 0-cell, 2 n-cells, and 1 $2n$-cell) or $|n - m| \geq 2$ and $n, m \geq 2$ (1 each of 0-, n-, m-, and $(n + m)$-cells).

More involved computations require a better understanding of what the boundary maps

$$H_k(X^{(k)}, X^{(k-1)}) \xrightarrow{d_n} H_{k-1}(X^{(k-1)}, X^{(k-2)})$$

are. These groups have as bases, essentially, the k-cells $\{e^n_\alpha\}$ and $(k-1)$-cells $\{e^{n-1}_\beta\}$ of X. In terms of these bases, letting $\varphi : D^n \to X^{(n)}$ be the characteristic map of e^n_α, $d_n(e^n_\alpha) = \sum n_{\alpha\beta} e^{n-1}_\beta$, where $n_{\alpha\beta}$ counts how many times the attaching map $f = \varphi|_{\partial D^n} : S^{n-1} \to X^{(n-1)}$ of e^n_α "passes over" e^{n-1}_β, in the following sense: taking the composition

$$\begin{aligned}\mathbb{Z} = \widetilde{H}_{n-1}(S^{n-1}) &\xrightarrow{f_*} \widetilde{H}_{n-1}(X^{(n-1)}) \xrightarrow{p_*} H_{n-1}(X^{(n-1)}, X^{(n-2)}) \\ &\cong \widetilde{H}_{n-1}(X^{(n-1)}/X^{(n-2)}) \cong \widetilde{H}_{n-1}(\vee_\beta S^{n-1}) \cong \oplus_\beta \widetilde{H}_{n-1}(S^{n-1}) = \oplus_\beta \mathbb{Z} \xrightarrow{proj_\beta} \mathbb{Z}\end{aligned}$$

sends 1 to $n_{\alpha\beta}$. (We omit the proof, but what else could it be....?)

$n_{\alpha\beta}$ is, then, what 1 gets sent to under the map on $\widetilde{H}_{n-1}$ induced by the map

$$S^{n-1} \xrightarrow{f} X^{(n-1)} \to X^{(n-1)}/X^{(n-2)} \cong \vee_\beta S^{n-1} \xrightarrow{p_\beta} S^{n-1}$$

which is, at least in principal, computable, given enough information about the attaching maps of our CW-complex X. This number $n_{\alpha\beta}$ is called the *degree* of the map $S^{n-1} \to S^{n-1}$. [It is in fact true that maps $f : S^n \to S^n$ are <u>determined</u> up to homotopy by their degree, but we will (probably) not prove this.] For example, for a homeomorphism $g : S^{n-1} \to S^{n-1}$, its degree is either 1 or -1 (since the induced map on $\widetilde{H}_{n-1}$ is an isomorphism).