Lifting properties:

Covering spaces of a (suitably nice) space X have a very close relationship to $\pi_1(X, x_0)$.

Homotopy Lifting Property: If $p : \widetilde{X} \to X$ is a covering map, $H : Y \times I \to X$ is a homotopy, $H(y, 0) = f(y)$, and $\widetilde{f} : Y \to \widetilde{X}$ is a *lift* of f (i.e., $p \circ \widetilde{f} = f$), then there is a unique lift $\widetilde{H}$ of H with $\widetilde{H}(y, 0) = \widetilde{f}(y)$.

In particular, applying this property in the case $Y = \{*\}$, where a homotopy $H : \{*\} \times I \to X$ is really just a a path $\gamma : I \to X$, we have the
Path Lifting Property: Given a covering map $p : \widetilde{X} \to X$, a path $\gamma : I \to X$ with $\gamma(0) = x_0$, and a point $\widetilde{x}_0 \in p^{-1}(x_0)$, there is a unique path $\widetilde{\gamma}$ lifting γ with $\widetilde{\gamma}(0) = \widetilde{x}_0$.
An immediate consequence:

If $p : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering map, then the induced homomorphism $p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective.

Proof: $\gamma : (I, \partial I) \to (\widetilde{X}, \widetilde{x}_0)$ a loop with $p_*([\gamma]) = 1$ in $\pi_1(X, x_0)$. There is $H : (I \times I, \partial I \times I) \to (X, x_0)$ interpolating between $p \circ \gamma$ and the constant path. By homotopy lifting, there is a homotopy $\widetilde{H}$ from γ to the lift of the constant map at x_0. The vertical sides $s \mapsto \widetilde{H}(0, s), \widetilde{H}(1, s)$ are also lifts of the constant map, beginning from $\widetilde{H}(0, 0), \widetilde{H}(1, 0) = \gamma(0) = \gamma(1) = \widetilde{x}_0$, so are the constant map at $\widetilde{x}_0$. So the lift at the bottom is the constant map at $\widetilde{x}_0$. So $\widetilde{H}$ represents a null-homotopy of γ, so $[\gamma] = 1$ in $\pi_1(\widetilde{X}, \widetilde{x}_0)$.

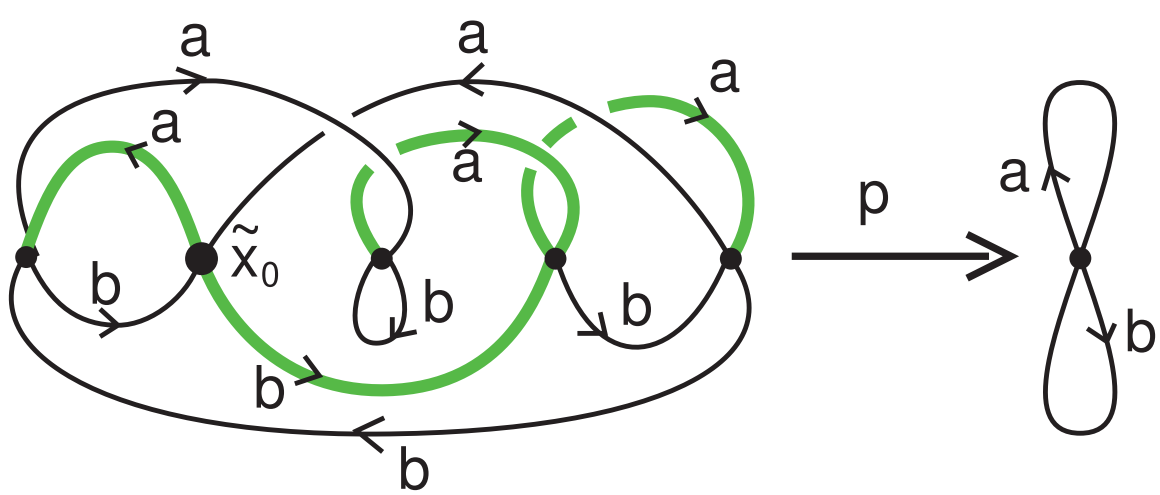
Even more, $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))) \subseteq \pi_1(X, x_0)$ is precisely the elements given by loops at x_0, whose lifts to paths starting at $\widetilde{x}_0$, are loops. If γ lifts to a loop $\widetilde{\gamma}$, then $p \circ \widetilde{\gamma} = \gamma$, so $p_*([\widetilde{\gamma}]) = [\gamma]$. If $p_*([\widetilde{\gamma}]) = [\gamma]$, then $\gamma \simeq p \circ \widetilde{\gamma}$ rel endpoints; the homotopy lifts to a homotopy b/w the lift of γ at $\widetilde{x}_0$ and the lift of $p \circ \widetilde{\gamma}$ at $\widetilde{x}_0$ (which is $\widetilde{\gamma}$, since $\widetilde{\gamma}(0) = \widetilde{x}_0$ and lifts are unique). So the lift of γ is a loop, as desired.

Proof of H.L.P.: lift maps a little bit at a time! Cover X by evenly covered open sets $\mathcal{U}_i$. For each fixed $y \in Y$, since I is compact and the sets $H^{-1}(\mathcal{U}_i)$ form an open cover of $Y \times I$, the Tube Lemma provides an open neighborhood $\mathcal{V}_y$ of y in Y and finitely many $p^{-1}\mathcal{U}_i$ which cover $\mathcal{V}_y \times I$.

To define $\widetilde{H}(y,t) = \widetilde{H}_y(t)$, cut $\{y\} \times I$ into pieces I_j, each mapping into some $\mathcal{U}_j$ under H. Starting from the left, we have (inductively) a lift $\widetilde{H}_y(t_j)$ of the left endpoint t_j of I_j to $\widetilde{X}$, and a homeo $h_j : \mathcal{U}_j \to$ the component of its inverse image of $\mathcal{U}_j$ containing $\widetilde{H}_y(t_j)$. Then define $\widetilde{H}_y$ on I_j to be $h_j \circ H_y$. By induction, $\widetilde{H}(y,t)$ is defined for all t (and y). This definition is independent of the partition $\{y\} \times I$, by the usual process of taking the union of the partitions, and noticing that the choice of h_j is unique. (If we change the open cover, we can compare using the intersections; the choice of h_j's will be the same.) $\widetilde{H}$ is a lift of H since $p \circ \widetilde{H} = p \circ (h_j \circ H) = (p \circ h_j) \circ H = I \circ H = H$. $\widetilde{H}$ is continuous since for y near y_0 we can use the same partitions and the same open cover (because of our tube lemma condition), which means that we use the same maps h_j to lift; the pasting lemma implies continuity.

So, for example, if we build a 5-sheeted cover of the bouquet of 2 circles, as before, (after choosing a maximal tree upstairs) we can read off the images of the generators of the fundamental group of the total space; we have labelled each edge by the generator it traces out downstairs, and for each edge outside of the maximal tree chosen, we read from basepoint out the tree to one end, across the edge, and then back to the basepoint in the tree. In our example, this gives:

$$< ab, aaab^{-1}, baba^{-1}, baa, ba^{-1}bab^{-1}, bba^{-1}b^{-1}| >$$



This is (from its construction) a copy of the free group on 6 letters, in the free group $F(a, b)$. In a similar way, by explicitly building a covering space, we find that the fundamental group of a closed surface of genus 3 is a subgroup of the fundamental group of the closed surface of genus 2.

The cardinality of a point inverse $p^{-1}(y)$ is, by the evenly covered property, constant on (small) open sets, so the set of points of x whose point inverses have any given cardinality is open. Consequently, if X is connected, this number is constant over all of X, and is called the number of *sheets* of the covering $p : \widetilde{X} \to X$.

The number of sheets of a covering map can also be determined from the fundamental groups:

Proposition: If X and $\widetilde{X}$ are path-connected, then the number of sheets of a covering map equals the index of the subgroup $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)$ in $G = \pi_1(X, x_0)$.

—bf Proof: Choose loops $\{g_\alpha = [\gamma_\alpha]\}$, one in each of the (right) cosets of H in G. Lift them to loops based at $\widetilde{x}_0$; they will have distinct endpoints. (If $\widetilde{\gamma}_1(1) = \widetilde{\gamma}_2(1)$, then by uniqueness of lifts, $\gamma_1 * \overline{\gamma_2}$ lifts to $\widetilde{\gamma}_1 * \overline{\widetilde{\gamma}_2}$, so it lifts to a loop, so $\gamma_1 * \overline{\gamma_2}$ represents an element of $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)$, so they are in the same coset.) Conversely, every point in $p^{-1}(x_0)$ is the endpoint of one of these lifts, since we can construct a path $\widetilde{\gamma}$ from $\widetilde{x}_0$ to any such point y, giving a loop $\gamma = p \circ \widetilde{\gamma}$ representing an element $g \in \pi_1(X, x_0)$. But then $g = hg_\alpha$ for some $h \in H$ and α, so γ is homotopic rel endpoints to $\eta * \gamma_\alpha$ for some loop η representing h. But then lifting these based at $\widetilde{x}_0$, by homotopy lifting, $\widetilde{\gamma}$ is homotopic rel endpoints to $\widetilde{\eta}$, which is a loop, followed by the lift $\widetilde{\gamma}_\alpha$ of γ_α starting at $\widetilde{x}_0$. So $\widetilde{\gamma}$ and $\widetilde{\gamma}_\alpha$ have the same value at 1.

Therefore, lifts of representatives of coset representatives of H in G give a 1-to-1 correspondence, given by $\widetilde{\gamma}(1)$, with $p^{-1}x_0$. In particular, they have the same cardinality.

The path lifting property (because $\pi([0,1],0)=\{1\}$) is a special case of the **lifting criterion**: If $p:(\widetilde{X},\widetilde{x}_0)\to(X,x_0)$ is a covering map, and $f:(Y,y_0)\to(X,x_0)$ is a map, where Y is path-connected and locally path-connected, then there is a unique lift $\widetilde{f}:(Y,y_0)\to(\widetilde{X},\widetilde{x}_0)$ of f (i.e., $f=p\circ\widetilde{f}$) $\Leftrightarrow$ $f_*(\pi_1(Y,y_0))\subseteq p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$.

If $\widetilde{f}$ exists, then $f_*=p_*\circ\widetilde{f}_*$, so $f_*(\pi_1(Y,y_0))=p_*(\widetilde{f}_*(\pi_1(Y,y_0)))\subseteq p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$.

Conversely, if $f_*(\pi_1(Y,y_0))\subseteq p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$, we will use path lifting to build the lift. Given $y\in Y$, choose a path γ in Y from y_0 to y and lift the path $f\circ\gamma$ in X to a path $\widetilde{f\circ\gamma}$ with $\widetilde{f\circ\gamma}(0)=\widetilde{x}_0$. Then define $\widetilde{f}(y)=\widetilde{f\circ\gamma}(1)$. <u>If</u> well-defined and continuous, this is our required lift, since $(p\circ\widetilde{f})(y)=p(\widetilde{f}(y))=p(\widetilde{f\circ\gamma}(1))=p\circ\widetilde{f\circ\gamma}(1)=(f\circ\gamma)(1)=f(\gamma(1))=f(y)$. For <u>well-defined</u>, if η is a path from y_0 to y, then $\gamma*\overline{\eta}$ is a loop, so $f\circ(\gamma*\overline{\eta})=(f\circ\gamma)*\overline{(f\circ\eta)}$ is a loop, giving an element of $f_*(\pi_1(Y,y_0))\subseteq p_*(\pi_1(\widetilde{X},\widetilde{x}_0))$, and so lifts to a loop based at $\widetilde{x}_0$. So $f\circ\gamma$ and $f\circ\eta$ lift, starting at $\widetilde{x}_0$, to have the same value at 1. So $\widetilde{f}$ is well-defined. <u>Continuity</u> comes from the evenly covered property of p. Given $y\in Y$, and a nbhd $\widetilde{\mathcal{U}}$ of $\widetilde{f}(y)$ in $\widetilde{X}$, we want a nbhd $\mathcal{V}$ of y with $\widetilde{f}(\mathcal{V})\subseteq\widetilde{\mathcal{U}}$. Choose an evenly covered nbhd $\mathcal{U}_y$ for $f(y)$, the sheet $\widetilde{\mathcal{U}}_y$ over $\mathcal{U}_y$ which contains $\widetilde{f}(y)$, and set $\mathcal{W}=\widetilde{\mathcal{U}}\cap\widetilde{\mathcal{U}}_y$. p is a homeo from $\mathcal{W}$ to the open set $p(\mathcal{W})\subseteq X$. Then if we set $\mathcal{V}'=f^{-1}(p(\mathcal{W}))$ this is open and contains y, and so contains a path-connected open nbhd $\mathcal{V}$ of y. Then for every point $z\in\mathcal{V}$ we compute $\widetilde{f}(z)$ by a path γ from y_0 to z which first goes to y and then, *in* $\mathcal{V}$, from y to z. Then by unique path lifting, since $f(\mathcal{V})\subseteq\mathcal{U}_y$, $f\circ\gamma$ lifts to the concatenation of a path from $\widetilde{x}_0$ to $\widetilde{f}(y)$ and a path *in* $\widetilde{\mathcal{U}}_y$ from $\widetilde{f}(y)$ to $\widetilde{f}(z)$. So $\widetilde{f}(z)\in\widetilde{\mathcal{U}}$.