CW complexes: The "right" spaces to do algebraic topology on.

The basic idea: CW complexes are built inductively, by gluing disks onto lower-dimensional strata. $X = \bigcup X^{(n)}$, where

$X^{(0)}$ = a disjoint union of points, and, inductively,

$X^{(n)}$ is built from $X^{(n-1)}$ by gluing n-disks D_i^n along their boundaries. That is we have $f_i : \partial D_i^n \to X^{(n-1)}$ and $X^{(n)} = X^{(n-1)} \cup (\coprod D_i^n)/\sim$ where $x \sim f_i(x)$ for all $x \in \partial D_i^n$. We have (natural) inclusions $X^{(n-1)} \subseteq X^{(n)}$, and $X = \bigcup X^{(n)}$ is given the *weak topology*; that is, $C \subseteq X$ is closed $\Leftrightarrow$ $C \cap X^{(n)}$ is closed for all n.
(Note: this is reasonable; $X^{(n-1)}$ is closed in $X^{(n)}$ for all n.)

Each disk D_i^n has a *characteristic map* $\phi_i : D_i^n \to X$ given by
$D_i^n \to X^{(n-1)} \cup (\coprod D_i^n) \to X^{(n)} \subseteq X$.
$f : X \to Y$ is cts $\Leftrightarrow$ $f \circ \phi_i : D_i^n \to X \to Y$ is cts for all D_i^n.
(This is a consequence of using the weak topology.)

A *CW pair* (X, A) is a CW complex X and a *subcomplex* A, which is a subset which is a union of images of cells, so it is a CW complex in its own right. We can induce CW structures under many standard constructions; e.g., if (X, A) is a CW pair, then X/A admits a CW structure whose cells are $[A]$ and the cells of X not in A. We can glue two CW complexes X, Y along isomorphic subcomplexes $A \subseteq X, Y$, yielding $X \cup_A Y$.

"CW"=closure finiteness, weak topology

Perhaps the most important property of CW complexes (for algebraic topology, anyway) is the *homotopy extension property*; given a CW pair (X, A), a map $f : X \to Y$, and a homotopy $H : A \times I \to Y$ such that $H|_{A\times 0} = f|_A$, there is a homotopy (extension) $K : X \times I \to Y$ with $K|_{A\times I} = H$. This is because $B = X \times \{0\} \cup A \times I$ is a retract of $X \times I$; K is the composition of this retraction and the "obvious" map from B to Y.

To build the retraction, we do it one cell of X at a time. The idea is that the retraction is defined on the cells of A (it's the identity), so look at cells of X not in A. Working our way up in dimension, we can assume the the retraction r_{n-1} is defined on (the image of) $\partial D^n \times I$, i.e., on $X^{(n-1)} \times I$. But $D^n \times I$ (strong deformation) retracts onto $D^n \times 0 \cup \partial D^n \times I$; composition of r_{n-1} with this retraction extends the retraction over $\phi(D^n) \times I$, and so over $X^{(n)} \times I$..

This, for example, lets us show that if (X, A) is a CW pair and A is contractible, then $X/A \simeq X$. This is because the composition $A \to * \to A$ is homotopic to the identity I_A, via some map $H : A \times I \to A$, with $H|_{A\times 0} = I_A$. Thinking of H as mapping into X, then together with the map $I_X : X \to X$ the HEP provides a map $K : X \times I \to X$ with $K_0 = I_X$ and $K_1(A) = *$. Setting $K_1 = g : X \to X$, it induces a map $h : X/A \to X$. This is a homotopy inverse of the projection $p : X \to X/A$:

$h \circ p = g \simeq I_X$ via K, and $p \circ h : X/A \to X/A$ is homotopic to $I_{X/A}$ since $K_t(A) \subseteq A$ for every t, so induces a map $\overline{K}_t : X/A \to X/A$, giving a homotopy between $\overline{K}_0 = \overline{I_X} = I_{X/A}$ and $\overline{K}_1 = \overline{g} = p \circ h$.