Group presentations:

$\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(D^2) \cong \{1\}$. Spheres and disks form the basic building blocks for many important spaces. The Seifert-van Kampen Theorem describes how to construct $\pi_1(X)$ for $X = A \cup B$ from $\pi_1(A)$, $\pi_1(B)$, and $\pi_1(A \cap B)$. This allows us to compute the fundamental groups of increasingly sophisticated spaces. To do so, we need the language of presentations.

The basic idea: a group G is *generated by a set $\Sigma \subseteq G$* if every element of G can be expressed as a finite product of elements of Σ (and their inverses). A product of elements of $\Sigma^{\pm 1}$ which equals 1 in the group is a *relator*; a set of relators R form a *presentation* $\langle \Sigma | R \rangle$ for G if all of the relators in G are "consequences" of those in R.

Free groups: Σ = a set; a *reduced word* on Σ is a (formal) product $a_1^{\epsilon_1} \cdots a_n^{\epsilon_n}$ with $a_i \in \Sigma$ and $\epsilon_i = \pm 1$, and either $a_i \neq a_{i+1}$ or $\epsilon_i \neq -\epsilon_{i+1}$ for every i. (I.e., no $aa^{-1}, a^{-1}a$ in the product.)

The free group $F(\Sigma)$ = the set of reduced words, with multiplication = concatenation followed by *reduction*: remove all occurances of $aa^{-1}, a^{-1}a$.

identity element = the empty word, $(a_1^{\epsilon_1} \cdots a_n^{\epsilon_n})^{-1} = a_n^{-\epsilon_n} \cdots a_1^{-\epsilon_1}$. $F(\Sigma)$ is generated by Σ, with no relations among the generators other than the "obvious" ones.

Important property of free groups: any function $f : \Sigma \to G$, G a group, extends uniquely to a homomorphism $\phi : F(\Sigma) \to G$.

If $R \subseteq F(\Sigma)$, then $< R >^N$ = normal subgroup generated by R =
$\{\prod_{i=1}^{n} g_i r_i g_i^{-1} : n \in \mathbb{N}_0, g_i \in F(\Sigma), r_i \in R\}$ = smallest normal subgroup containing R.

$F(\Sigma)/ < R >^N$ = the group with *presentation* $< \Sigma | R >$; it is the largest quotient of $F(\Sigma)$ in which the elements of R are the identity.

Every group has a presentation: $G \cong F(G)/ < gh(gh)^{-1} : g, h \in G >^N$
where (gh) is interpreted as a single letter in G.

If $G_1 =< \Sigma_1 | R_1 >$ and $G_2 =< \Sigma_2 | R_2 >$, then their *free product* $G_1 * G_2 =< \Sigma_1 \coprod \Sigma_2 | R_1 \cup R_2 >$ (Σ_1, Σ_2 must be treated as (formally) disjoint). Each $g \in G_1 * G_2$ is $g = g_1 \cdots g_n$ where the g_i alternate from G_1, G_2 (uniquely). G_1, G_2 are subgroups of $G_1 * G_2$ in the obvious way. Important property of free products: any pair of homoms $\phi_i : G_i \to G$ extends uniquely to a homom $\phi : G_1 * G_2 \to G$.

Gluing groups: given groups G_1, G_2, with subgroups H_1, H_2 that are isomorphic $H_1 \cong H_2$, we can "glue" G_1 and G_2 together along their "common" subgroup. More generally, given a group H and homomorphisms $\phi_1 : H \to G_i$, we can build the largest group "generated" by G_1 and G_2, in which $\phi_1(h) = \phi_2(h)$ for all $h \in H$.

Starting with $G_1 * G_2$ (to get the first part), we then take a quotient to insure that $\phi_1(h)(\phi_2(h))^{-1} = 1$ for every h. Using presentations $G_1 =< \Sigma_1 | R_1 >$, $G_2 =< \Sigma_2 | R_2 >$, quotienting out by as little as possible, we have

$$G = (G_1 * G_2) / < \phi_1(h)(\phi_2(h))^{-1} : h \in H >^N = \\ < \Sigma_1 \coprod \Sigma_2 | R_1 \cup R_2 \cup \{\phi_1(h)(\phi_2(h))^{-1} : h \in H\} >$$

$G == G_1 *_H G_2$ is the *largest* group generated by G_1 and G_2 in which $\phi_1(h) = \phi_2(h)$ for all $h \in H$, and is called the *amalgamated free product* or *free product with amalgamation (over H)* or *pushout over H*.

Important special cases : $G *_H \{1\} = G / < \phi(H) >^N =< \Sigma | R \cup \phi(H) >$, and $G_1 *_{\{1\}} G_2 \cong G_1 * G_2$