The Fundamental Theorem of Algebra:

Every non-constant polynomial has a complex root: for every $f(z) = a_n z^n + \cdots + a_0$ with $n \geq 1$ and $a_n \neq 0$, $a_i \in \mathbb{C}$, there is a $z_0 \in \mathbb{C}$ with $f(z_0) = 0$.

Proof: Thinking of $\mathbb{C} = \mathbb{R}^2$, if not, then f is a map $\mathbb{R}^2 \to \mathbb{R}^2 \setminus \{0\}$. We can divide through by a_n without affecting this, and assume that f is monic.

Setting $\gamma_m(t) = f(m\cos(2\pi t), m\sin(2\pi t))$, then $\gamma_m : S^1 \to \mathbb{R}^2 \setminus \{0\}$ extends to a map $\Gamma_m : D^2 \to \mathbb{R}^2 \setminus \{0\}$, as $\Gamma_n(x) = f(mx)$, so γ_m is null-homotopic for all m.

But $\mathbb{R}^2 \setminus \{0\}$ def. retracts to the unit circle (by $r(z) = z/|z|$), so $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{Z}$, and by the above all of the $[\gamma_m]$ represent 0 in $\mathbb{Z}$, and so $r_*[\gamma_m] = [r \circ \gamma_m] = 0$, as well. But for large m we can compute $w(r \circ \gamma_m) = n$; since $n \geq 1$, this is a contradiction.

$\gamma_m(t) = f(me^{2\pi it}) = m^n(e^{2\pi nit} + \frac{a_{n-1}}{m}e^{2\pi(n-1)it} + \cdots + \frac{a_0}{m^n}) = m^n(e^{2\pi nit} + R(m,t))$, so $r \circ \gamma_m(t) = (e^{2\pi nit} + R(m,t))/|e^{2\pi nit} + R(m,t)|$. But as $m \to \infty$, $R(m,t) \to 0$ uniformly in t; $|R(m,t)| = |\frac{a_{n-1}}{m}e^{2\pi(n-1)it} + \cdots + \frac{a_0}{m^n}| \leq \frac{|a_{n-1}|}{m} + \cdots + \frac{|a_0|}{m^n} \to 0$.

So for large enough m $|R(m,t)| < \frac{1}{2}$ for all t,
and then for every $s \in I$, $|e^{2\pi nit} + sR(m,t)| \neq 0$, since
$|e^{2\pi nit} + sR(m,t)| \geq |e^{2\pi nit}| - s|R(m,t)| \geq |e^{2\pi nit}| - |R(m,t)| \geq \frac{1}{2}$.

Then the homotopy $H(t,s) = (e^{2\pi nit} + sR(m,t))/|e^{2\pi nit} + sR(m,t)|$ is well-defined and continuous, $H : I \times I \to S^1$, and defines a homotopy from $\alpha : t \mapsto e^{2\pi nit}$ (at $s = 0$) to $r \circ \gamma_m$ (at $s = 1$). Since $w(\alpha) = n$, for large enough m, $w(r \circ \gamma_m) = n$.

This contradiction implies that f must have a root, as desired.