**The homotopy realm**

$\pi_1(X, x_0)$ uses loops $\gamma$, but treats two the same if they are (based) homotopic.
If $f, g : X \to Y$ are homotopic maps, then $f \circ \gamma \simeq g \circ \gamma$, so we expect homotopic maps to descend to the "same" maps on $\pi_1$. This is almost true; you need to adjust for the change-of-basepoint map $\widehat{\alpha_H}$ for $\alpha_H(t) = H(x_0, t)$, since the homotopy will drag the basepoint along this path. So $\pi_1$ is fairly insensitive to homotopies. This motivates:

Spaces $X, Y$ are *homotopy equivalent* if there are $f : X \to Y$ and $g : Y \to X$ so that $f \circ g \simeq I_Y$ and $g \circ f \simeq I_X$ (via $H$ and $K$). If $\alpha_H, \alpha_K$ are the traces of the basepoints, then $f_* \circ g_* = \widehat{\alpha_H}$ and $g_* \circ f_* = \widehat{\alpha_K}$ are isomorphisms, so $f_*, g_*$ are isomorphisms. So homotopy equivalent spaces have isomorphic fundamental groups.

A special case: $A \subseteq X$, $r : X \to A$ is a retraction ($r \circ \iota = I_A$), and $\iota \circ r \simeq I_X$ (via a homotopy $H$). $A$ is then called a *reformation retract* of $X$. If $H$ fixes $A$ (i.e., $H(a, t) = a$ for all $a \in A$), then $A$ is a *strong deformation retract* of $X$. In both cases, $\iota_* : \pi_1(A) \to \pi_1(X)$ is an isomorphism. A space is *contractible* if it deformation retracts to a point (e.g., $I, D^n, \mathbb{R}^n$). Contractible spaces have trivial fundamental group. Path-connected spaces $X$ with $\pi_1(X) = \{1\}$ is called *simply connected.*

A loop $\gamma : (I, \partial I) \to (X,_0)$ induces a map $\gamma_1 : S^1 \cong I/0, 1 \to X$ . Elements of $\pi_1(X, x_0)$ can be thought of as homotopy (of pairs) classes of maps $(S^1, 1) \to (X, x_0)$.
From this perspective, $\gamma : S^1 \to X$ represents the identity in $\pi_1(X) \Leftrightarrow$ $\gamma$ extends to a map $\Gamma : D^2 \to X$. (The extension is $\Gamma(re^{2\pi i\theta}) = H(\theta, 1 - r)$.)

Similarly, two paths $\alpha, \beta : I \to X$ joining the same pair of points $x_0, x_1 \in X$ are homotopic rel endpoints (i.e., the maps $(I\partial I) \to (X, \{x_0, x_1\})$ are homotopic as maps of pairs) $\Leftrightarrow$ the loop $\alpha * \overline{\beta}$ is trivial in $\pi_1(X, x_0)$. So, for example, in a contractible space, any two paths between the same two points are homotopic rel endpoints.