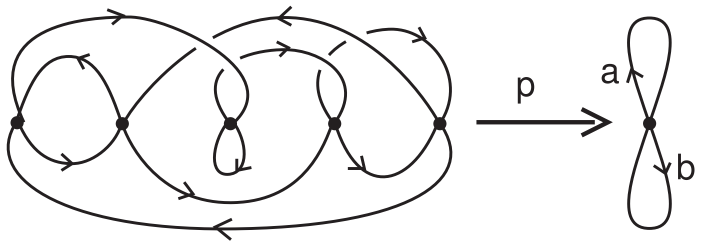
Math 872 Algebraic Topology

Running lecture notes

Covering spaces: We can motivate our next topic by looking more closely at one of our examples above. The projective plane $\mathbb{R}P^2$ has $\pi_1 = \mathbb{Z}_2$. It is also the quotient of the simply-connected space S^2 by the antipodal map, which, together with the identity map, forms a group of homeomorphisms of S^2 which is isomorphic to $\mathbb{Z}_2$. The fact that $\mathbb{Z}_2$ has this dual role to play in describing $\mathbb{R}P^2$ is no accident; codifying this relationship requires the notion of a covering space.

The quotient map $q : S^2 \to \mathbb{R}P^2$ is an example of a *covering map.* A map $p : E \to B$ is called a covering map if for every point $x \in B$, there is a neighborhood $\mathcal{U}$ of x (an *evenly covered neighborhood*) so that $p^{-1}(\mathcal{U})$ is a disjoint union $\mathcal{U}_\alpha$ of open sets in E, each mapped homeomorphically onto $\mathcal{U}$ by (the restriction of) p . B is called the *base space* of the covering; E is called the *total space.* The quotient map q is an example; (the image of) the complement of a great circle in S^2 will be an evenly covered neighborhood of any point it contains. The disjoint union of 43 copies of a space, each mapping homeomorphically to a single copy, is an example of a *trivial covering.* As a last example, we have the famous exponential map $p : \mathbb{R} \to S^1$ given by $t \mapsto e^{2\pi i t} = (\cos(2\pi t), \sin(2\pi t))$. The image of any interval (a, b) of length less than 1 will have inverse image the disjoint union of the intervals $(a + n, b + n)$ for $n \in \mathbb{Z}$.

OK, maybe not the last. We can build many finite-sheeted (every point inverse is finite) coverings of a bouquet of two circles, say, by assembling n points over the vertex, and then, on either side, connecting the points by n (oriented) arcs, one each going in and out of each vertex. By choosing orientations on each 1-cell of the bouquet, we can build a covering map by sending the vertices above to the vertex, and the arcs to the one cells, homeomorphically, respecting the orientations. We can build infinite-sheeted coverings in much the same way.



Covering spaces of a (suitably nice) space X have a very close relationship to $\pi_1(X, x_0)$. The basis for this relationship is the

Homotopy Lifting Property: If $p : \widetilde{X} \to X$ is a covering map, $H : Y \times I \to X$ is a homotopy, $H(y, 0) = f(y)$, and $\widetilde{f} : Y \to \widetilde{X}$ is a *lift* of f (i.e., $p \circ \widetilde{f} = f$), then there is a unique lift $\widetilde{H}$ of H with $\widetilde{H}(y, 0) = \widetilde{f}(y)$.

The **proof** of this we will defer to next time, to give us sufficient time to ensure we finish it!

In particular, applying this property in the case $Y = \{*\}$, where a homotopy $H : \{*\} \times I \to X$ is really just a a path $\gamma : I \to X$, we have the **Path Lifting Property**: "given a covering map $p : \widetilde{X} \to X$, a path $\gamma : I \to X$ with $\gamma(0) = x_0$, and a point $\widetilde{x}_0 \in p^{-1}(x_0)$, there is a unique path $\widetilde{\gamma}$ lifting γ with $\widetilde{\gamma}(0) = \widetilde{x}_0$." One of the immediate consequences of this is one of the cornerstones of covering space theory:

If $p : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering map, then the induced homomorphism $p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective.

Proof: Suppose $\gamma : (I, \partial I) \to (\widetilde{X}, \widetilde{x}_0)$ is a loop $p_*([\gamma]) = 1$ in $\pi_1(X, x_0)$. So there is a homotopy $H : (I \times I, \partial I \times I) \to (X, x_0)$ between $p \circ \gamma$ and the constant path. By homotopy lifting, there is a homotopy $\widetilde{H}$ from γ to the lift of the constant map at x_0. The vertical sides $s \mapsto \widetilde{H}(0, s), \widetilde{H}(1, s)$ are also lifts of the constant map, beginning from $\widetilde{H}(0, 0), \widetilde{H}(1, 0) = \gamma(0) = \gamma(1) = \widetilde{x}_0$, so are the constant map at $\widetilde{x}_0$. Consequently, the lift at the bottom is the constant map at $\widetilde{x}_0$. So $\widetilde{H}$ represents a null-homotopy of γ, so $[\gamma] = 1$ in $\pi_1(\widetilde{X}, \widetilde{x}_0)$. So $\pi_1(\widetilde{X}, \widetilde{x}_0) = \{1\}$.

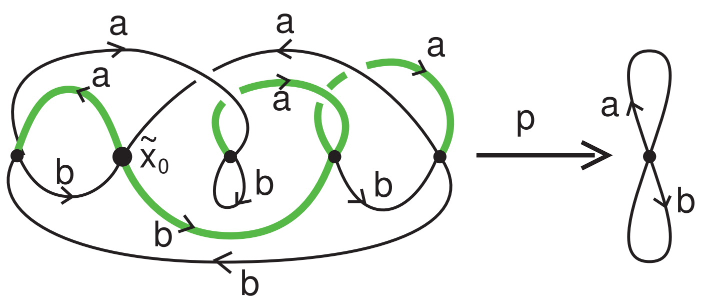
Even more, the image $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))) \subseteq \pi_1(X, x_0)$ is precisely the elements whose representatives are loops at x_0, which when lifted to paths starting at $\widetilde{x}_0$), are loops. For if γ lifts to a loop $\widetilde{\gamma}$, then $p \circ \widetilde{\gamma} = \gamma$, so $p_*([\widetilde{\gamma}]) = [\gamma]$. Conversely, if $p_*([\widetilde{\gamma}]) = [\gamma]$, then γ and $p \circ \widetilde{\gamma}$ are homotopic rel endpoints, and so the homotopy lifts to a homotopy rel endpoints between the lift of γ at $\widetilde{x}_0$, and the lift of $p \circ \widetilde{\gamma}$ at $\widetilde{x}_0$ (which is $\widetilde{\gamma}$, since $\widetilde{\gamma}(0) = \widetilde{x}_0$ and lifts are unique). So the lift of γ is a loop, as desired.

The **proof** of the homotopy lifting property follows a pattern that we will become very familiar with: we lift maps a little bit at a time. For every $x \in X$ there is an open set $\mathcal{U}_x$ evenly covered by p . For each fixed $y \in Y$, since I is compact and the sets $H^{-1}(\mathcal{U}_x)$ form an open cover of $Y \times I$, then since I is compact, the Tube Lemma provides an open neighborhood $\mathcal{V}$ of y in Y and finitely many $p^{-1}\mathcal{U}_x$ whose union covers $\mathcal{V} \times I$.

To define $\widetilde{H}(y, t)$, we (using a Lebesgue number argument) cut the interval $\{y\} \times I$ into finitely many pieces, the ith mapping into $\mathcal{U}_{x_i}$ under H. $\widetilde{f}(y)$ is in one of the evenly covered sets $\mathcal{U}_{x_1\alpha_1}$, and the restricted map $p^{-1} : \mathcal{U}_{x_1} \to \mathcal{U}_{x_1\alpha_1}$ following H restricted to the first interval lifts H along the first interval to a map we will call $\widetilde{H}$. We then have lifted H at the end of the first interval = the beginning of the second, and we continue as before. In this way we can define $\widetilde{H}$ for all (y, t) . To show that this is independent of the choices we have made along the way, we imagine two ways of cutting up the interval $\{y\} \times I$ using evenly covered neighborhoods $\mathcal{U}_{x_i}$ and $\mathcal{V}_{w_j}$, and take intersections of both sets of intervals to get a common refinement of both sets, covered by the intersections $\mathcal{U}_{x_i} \cap \mathcal{V}_{w_j}$, and imagine building $\widetilde{H}$ using the refinement. At the start, at $\widetilde{f}(y)$, we are in $\mathcal{U}_{x_1\alpha_1} \cap \mathcal{V}_{w_1\beta_1}$. Because at the start of the lift $(y, 0)$ we lift to the same point, and p^{-1} restricted to this intersection agrees with p^{-1} restricted to each of the two pieces, we get the same lift acroos the first refined subinterval. This process repeats itself across all of the subintervals, showing that the lift is independent of the choices made. This also shows that the lift is unique; once we have decided what $\widetilde{H}(y, 0)$, the rest of the values of the $\widetilde{H}$ are determined by the requirement of being a lift. also, once we know the map is well-defined, we can see that it is continuous, since for any y, we can make the same choices across the entire open set V given by the Tube Lemma, and find that $\widetilde{H}$, restricted to $\mathcal{V} \times (a_i - \delta, b_i + \delta)$ (for a small delta; we could wiggle the endpoints in the construction without changing the resulting function, by its well-definedness) is H estricted to this set followed by p^{-1} restriced in domain and range, so this composition is continuous. So $\widetilde{H}$ is locally continuous, hence continuous.

So, for example, if we build a 5-sheeted cover of the bouquet of 2 circles, as before, (after choosing a maximal tree upstairs) we can read off the images of the generators of the fundamental group of the total space; we have labelled each ede by the ereator it traces out downstairs, and for each ede outside of the maximal tree chosen, we read from basepoint out the tree to one end, across the edge, and then back to the basepoint in the tree. In our example, this gives:

$$< ab, aaab^{-1}, baba^{-1}, baa, ba^{-1}bab^{-1}, bba^{-1}b^{-1}| >$$



This is (from its construction) a copy of the free group on 6 letters, in the free group $F(a, b)$. In a similar way, by explicitly building a covering space, we find that the fundamental group of a closed surface of genus 3 is a subgroup of the fundamental group of the closed surface of genus 2.

The cardinality of a point inverse $p^{-1}(y)$ is, by the evenly covered property, constant on (small) open sets, so the set of points of x whose point inverses have any given cardinality is open. Consequently, if X is connected, this number is constant over all of X, and is called the number of *sheets* of the covering $p : \widetilde{X} \to X$.

The number of sheets of a covering map can also be determined from the fundamental groups:

Proposition: If X and $\widetilde{X}$ are path-connected, then the number of sheets of a covering map equals the index of the subgroup $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)$ in $G = \pi_1(X, x_0)$.

To see this, choose loops $\{\gamma_\alpha\}$ representing representatives $\{g_\alpha\}$ of each of the (right) cosets of H in G. Then if we lift each of them to loops based at $\widetilde{x}_0$, they will have distinct endpoints; if $\widetilde{\gamma}_1(1) = \widetilde{\gamma}_2(1)$, then by uniqueness of lifts, $\gamma_1 * \overline{\gamma_2}$ lifts to $\widetilde{\gamma}_1 * \overline{\widetilde{\gamma}_2}$, so it lifts to a loop, so $\gamma_1 * \overline{\gamma_2}$ represents an element of $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)$, so $g_1 = g_2$. Conversely, every point in $p^{-1}(x_0)$ is the endpoint of on of these lifts, since we can construct a path $\widetilde{\gamma}$ from $\widetilde{x}_0$ to any such point y, giving a loop $\gamma = p \circ \widetilde{\gamma}$ representing an element $g \in \pi_1(X, x_0)$. But then $g = hg_\alpha$ for some $h \in H$ and α, so γ is homotopic rel endpoints to $\eta * \gamma_\alpha$ for some loop η representing h. But then lifting these based at $\widetilde{x}_0$, by hmotopy lifting, $\widetilde{\gamma}$ is homotopic rel endpoints to $\widetilde{\eta}$, which is a loop, followed by the lift $\widetilde{\gamma}_\alpha$ of γ_α starting at $\widetilde{x}_0$. So $\widetilde{\gamma}$ and $\widetilde{\gamma}_\alpha$ have the same value at 1.

Therefore, lifts of representatives of coset representatives of H in G give a 1-to-1 correspondence, given by $\widetilde{\gamma}(1)$, with $p^{-1}x_0$. In particular, they have the same cardinality.

The path lifting property (because $\pi([0,1],0) = \{1\}$) is actually a special case of a more general **lifting criterion**: If $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering map, and $f : (Y, y_0) \to (X, x_0)$ is a map, where Y is path-connected and locally path-connected, then there is a lift $\widetilde{f} : (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ of f (i.e., $f = p \circ \widetilde{f}$) $\Leftrightarrow f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. Furthermore, two lifts of f which agree at a single point are equal.

If the lift exists, then $f = p \circ \widetilde{f}$ implies $f_* = p_* \circ \widetilde{f}_*$, so $f_*(\pi_1(Y, y_0)) = p_*(\widetilde{f}_*(\pi_1(Y, y_0))) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$, as desired. Conversely, if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$, then we wish to build the lift of f. Not wishing to waste our work on the special case, we will *use* path lifting to do it! Given $y \in Y$, choose a path γ in Y from y_0 to y and use path lifting in X to lift the path $f \circ \gamma$ to a path $\widetilde{f \circ \gamma}$ with $\widetilde{f \circ \gamma}(0) = \widetilde{x}_0$. Then define $\widetilde{f}(y) = \widetilde{f \circ \gamma}(1)$. Provided we show that this is well-defined and continuous, it is our required lift, since $(p \circ \widetilde{f})(y) = p(\widetilde{f}(y)) = p(\widetilde{f \circ \gamma}(1)) = p \circ \widetilde{f \circ \gamma})(1) = (f \circ \gamma)(1) = f(\gamma(1)) = f(y)$. To show that it is well-defined, if η is any other path from y_0 to y, then $\gamma * \overline{\eta}$ is a loop in Y, so $f \circ (\gamma * \overline{\eta}) = (f \circ \gamma) * \overline{(f \circ \eta)}$ is a loop in X representing an element of $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$, and so lifts to a loop in $\widetilde{X}$ based at $\widetilde{x}_0$. Consequently, as before, $f \circ \gamma$ and $f \circ \eta$ lift to paths starting at $\widetilde{x}_0$ with the same value at 1. So $\widetilde{f}$ is well-defined. To show that $\widetilde{f}$ is continuous, we use the evenly covered property of p. Given $y \in Y$, and a neighborhood $\widetilde{\mathcal{U}}$ of $\widetilde{f}(y)$ in $\widetilde{X}$, we wish to find a nbhd $\mathcal{V}$ of y with $\widetilde{f}(\mathcal{V}) \subseteq \widetilde{\mathcal{U}}$. Choosing an evenly covered neighborhood $\mathcal{U}_y$ for $f(y)$, choose the sheet $\widetilde{\mathcal{U}}_y$ over $\mathcal{U}_y$ which contains $\widetilde{f}(y)$, and set $\mathcal{W} = \widetilde{\mathcal{U}} \cap \widetilde{\mathcal{U}}_y$. This is open in $\widetilde{X}$, and p is a homeomorphism from this set to the open set $p(\mathcal{W}) \subseteq X$. Then if we set $\mathcal{V}' = f^{-1}(p(\mathcal{W})$ this is an open set containing y, and so contains a path-connected open set $\mathcal{V}$ containing y. Then is for every point $z \in \mathcal{V}$ we build a path γ from y_0 to z by concatenating a path from y_0 to y with a path *in* $\mathcal{V}$ from y to z, then by unique path lifting, since $f(\mathcal{V} \subseteq \mathcal{U}_y$, $f \circ \gamma$ lifts to the concatenation of a path from $\widetilde{x}_0$ to $\widetilde{f}(y)$ and a path *in* $\widetilde{\mathcal{U}}_y$ from $\widetilde{f}(y)$ to $\widetilde{f}(z)$. So $\widetilde{f}(z) \in \widetilde{\mathcal{U}}$.

Because $\widetilde{f}$ is built by lifting paths, and path lifting is unique, the last statement of the proposition follows.

Universal covering spaces: As we shall see, a particularly important covering space to identify is one which is simply connected. One thing we can see from the lifting crierion is that such a covering is essentially unique:

If X is locally path-connected, and has two connected, simply connected coverings $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$, then choosing basepoints $x_i, i = 0, 1, 2$, since $p_{1*}(\pi_1(X_1, x_1)) = p_{2*}(\pi_1(X_2, x_2)) = \{1\} \subseteq \pi_1(X, x_0)$, the lifting criterion with each projection playing the role of f in turn gives us maps $\widetilde{p}_1 : (X_1, x_1) \to (X_2, x_2)$ and $\widetilde{p}_2 : (X_2, x_2) \to (X_1, x_1)$ with $p_2 \circ \widetilde{p}_1 = p_1$ and $p_1 \circ \widetilde{p}_2 = p_2$. Consequently, $p_2 \circ \widetilde{p}_1 \circ \widetilde{p}_2 = p_1 \circ \widetilde{p}_2 = p_2$ and similarly, $p_1 \circ \widetilde{p}_2 \circ \widetilde{p}_1 = p_2 \circ \widetilde{p}_1 = p_1$. So $\widetilde{p}_1 \circ \widetilde{p}_2 : (X_2, x_2) \to (X_2, x_2)$, for example, is a lift of p_2 to the covering map p_2. But so is the identity map! By uniqueness, therefore, $\widetilde{p}_1 \circ \widetilde{p}_2 = Id$. Similarly, $\widetilde{p}_2 \circ \widetilde{p}_1 = Id$. So (X_1, x_1) and (X_2, x_2) are homeomorphic. So up to homeomorphism, a space can have only one connected, simply-connected covering space. It is known as the *universal covering* of the space X.

Not every (locall path-connected) space X has a universal covering; a (further) necessary condition is that X be *semi-locally simply connected*. The idea is that If $p : \widetilde{X} \to X$ is the universal cover, then for every point $x \in X$, we have an evenly-covered neighborhood $\mathcal{U}$ of x. The inclusion $i : \mathcal{U} \to X$, by definition, lifts to $\widetilde{X}$, so $i_*(\pi_1(\mathcal{U}, x)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}) = \{1\}$, so i_* is the trivial map. Consequently, every loop in $\mathcal{U}$ is null-homotopic in X. This is semi-local simple connectivity; every point has a neighborhood whose inclusion-induced homomorphism is trivial. Not all spaces have this property; the most famous is the Hawaiian earrings $X = \bigcup_n \{x \in \mathbb{R}^2 : ||x - (1/n, 0)|| = 1/n\}$. The point $(0,0)$ has no such neighborhood.

Building universal coverings: If a space X is path connected, locally path connected, and semi-locally simply connected (S-LSC), then it has a universal covering; we describe a general construction. The idea is that a covering space should have the path lifting and homotopy lifting properties, and the universal cover can be characterized as the only covering space for which *only* null-homotopic loops lift to loops. So we build a space and a map which <u>must</u> have these properties. We do this by making a space $\widetilde{X}$ whose points are (equivalence classes of) $[\gamma]$ based paths $\gamma : (I, 0) \to (X, x_0)$, where two paths are equivalent if they are homotopic rel endpoints! The projection map is $p([\gamma]) = \gamma(1)$. The S-LSCness is what guarantees that this is a covering map; choosing $x \in X$, a path γ_0 from x_0 to x, and a neighborhood $\mathcal{U}$ of x guaranteed by S-LSC, paths from x_0 to points in $\mathcal{U}$ are based equivalent to $\gamma * \gamma_0 * \eta$ where γ is a based loop at x_0 and η is a path in $\mathcal{U}$. But by simple connectivity, a path in $\mathcal{U}$ is determined up to homotopy by its endpoints, and so, with γ fixed, these paths are in one-to-one correspondence with $\mathcal{U}$. So $p^{-1}(\mathcal{U}$ is a disjoint union, indexed by $\pi_1(X, x_0)$, of sets in bijective correspondence with $\mathcal{U}$. The appropriate topology on $\widetilde{X}$, essentially given as a basis by triples $\gamma *, gamma_0, \mathcal{U}$ as above, make p a covering map. Note that the inverse image of the basepoint x_0 is the equivalence classes of <u>loops</u> at x_0, i.e., $\pi_1(X, x_0)$. A path γ lifts to the path of paths $[\gamma_t]$, where $\gamma_t(s) = \gamma(ts)$, and so the only loop in X which lifts to a loop in $\widetilde{X}$ has endpoint $[\gamma] = [c_{x_0}]$, i.e., $[\gamma] = 1$ in $\pi_1(X, x_0)$. This implies that $p_*(\pi_1(\widetilde{X}, [c_{x_0}])) = \{1\}$, so $\pi_1(\widetilde{X}, [c_{x_0}]) = \{1\}$. However, nobody in their right minds would go about building $\widetilde{X}$ in this way, in general! Before describing how to do it "right", though, we should perhaps see why we should want to?

One reason for the importance of the universal cover is that it gives us a unified approach to building <u>all</u> connected covering spaces of X. The basis for this is the *deck transformation group* of a covering space $p : \widetilde{X} \to X$; this is the set of all homeomorphisms $h : \widetilde{X} \to \widetilde{X}$ such that $p \circ h = p$. These homeomorphisms, by definition, permute each of the point inverses of

p. In fact, since h can be thought of as a lift of the projection p, by the lifting criterion h is determined by which point in the inverse image of the basepoint x_0 it takes the basepoint $\widetilde{x}_0$ of $\widetilde{X}$ to. A deck transformation sending $\widetilde{x}_0$ to $\widetilde{x}_1$ exists $\Leftrightarrow p_*(\pi_1(\widetilde{X}, \widetilde{x}_0) = p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)$ [we need one inclusion to give the map h, and the opposite inclusion to ensure it is a bijection (because its inverse exists)]. These two groups are in general *conjugate*, by the projection of a path from $\widetilde{x}_0$ to $\widetilde{x}_1$; this can be seen by following the change of basepoint isomorphism down into $G = \pi_1(X, x_0)$. As we have seen, paths in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$ are in 1-to-1 correspondence with the cosets of $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)$ in $p_*(\pi_1(X, x_0)$; so deck transformations are in 1-to-1 correspondence with cosets whose representatives conjugate H to itself. The set of such elements in G is called the *normalizer of H in G*, and denoted $N_G(H)$ or simply $N(H)$. The deck transformation group is therefore in 1-to-1 correspondence with the group $N(H)/H$ under $h \mapsto$ the coset represented by the projection of the path from $\widetilde{x}_0$ to $h(\widetilde{x}_0)$. And since h is essentially built by lifting paths, it follows quickly that this map is a homomorphism, hence an isomorphism.

In particular, applying this to the universal covering space $p : \widetilde{X} \to X$, since in this case $H = \{1\}$, so $N(H) = \pi_1(X, x_0)$, its deck transformation group is isomorphic to $\pi_1(X, x_0)$. For example, this gives the quickest possible proof that $\pi_1(S^1) \cong \mathbb{Z}$, since $\mathbb{R}$ is a contractible covering space, whose deck transformations are the translations by integer distances. Thus $\pi_1(X)$ acts on its universal cover as a group of homeomorphisms. And since this action is *simply transitive* on point inverses [there is exactly one (that's the simple part) deck transformation carrying any one point in a point inverse to any other one (that's the transitive part)], the quotient map from $\widetilde{X}$ to the orbits of this action <u>is</u> the projection map p. The evenly covered property of p implies that X does have the quotient topology under this action.

Every space it X the quotient of its universal cover (if it has one!) by its fundamental group $G = \pi_1(X, x_0)$, realized as the group of deck transformations. And the quotient map is the covering projection. So $X \cong \widetilde{X}/G$. In general, a quotient of a space Z by a group action G need not be a covering map; the action must be *properly discontinuous*, which means that for every point $z \in Z$, there is a neighborhood $\mathcal{U}$ of x so that $g \neq 1 \Rightarrow \mathcal{U} \cap g\mathcal{U} = \emptyset$ (the group action carries sufficiently small neighborhoods off of themselves). The evenly covered neighborhoods provide these for the universal cover. And conversely, the quotient of a space by a p.d. group action is a covering space.

But! Given $G = \pi_1(X, x_0)$ and its action on a universal cover $\widetilde{X}$, we can, instead of quotienting out by G, quotient out by any <u>subgroup</u> H of G, to build $X_H = \widetilde{X}/H$. This is a space with fundamental group H, having $\widetilde{X}$ as universal covering. And since the quotient (covering) map $p_G : \widetilde{X} \to X = \widetilde{X}/G$ factors through $\widetilde{X}/H$, we get an induced map $p_H : \widetilde{X}/H \to X$, which is a covering map; open sets with trivial inclusion-induced homomorphism lift homeomorphically to $\widetilde{X}$, hence homeomorphically to $\widetilde{X}/H$; taking lifts to each point inverse of $x \in X$ verifies the evenly covering property for p_H . So every subgroup of G is the fundamental group of a covering of X.

We can further refine this to give the *Galois correspondence.* Two covering spaces $p_1 : X_1 \to X$, $p_2 : X_2 \to X$ are *isomorphic* if there is a homeomorphism $h : X_1 \to X_2$

with $p_1 = p_2 \circ h$. Choosing basepoints x_1, x_2 mapping to $x_0 \in X$, this implies that, if $h(x_1) = x_2$, then $p_{1*}(\pi_1(X_1, x_1)) = p_{2*}(h_*(\pi_1(X_1, x_1))) = p_{2*}(\pi_1(X_2, x_2))$. On the other hand, our homeomorphism h need not take our chosen basepoints to one another; if $h(x_1) = x_3$, then $p_{1*}(\pi_1(X_1, x_1)) = p_{2*}(\pi_1(X_2, x_3))$. But $p_{2*}(\pi_1(X_2, x_2))$ and $p_{2*}(\pi_1(X_2, x_3))$ are isomorphic, via a change of basepoint isomorphism $\widehat{\eta}$, where η is a path in X_2 from x_2 to x_3. But such a path projects to X has a loop at x_0, and since the change of basepoint isomorphism is by "conjugating" by the path η, the resulting groups $p_{2*}(\pi_1(X_2, x_2))$ and $p_{2*}(\pi_1(X_2, x_3))$ are conjugate, by $p_2 \circ \eta$. So, without reference to basepoints, isomorphic coverings give, under projection, conjugate subgroups of $\pi_1(X, x_0)$. But conversely, given covering spaces X_1, X_2 whose subgroups $p_{1*}(\pi_1(X_1, x_1))$ and $p_{2*}(\pi_1(X_2, x_2))$ are conjugate, lifting a loop γ representing the conjugating element to a loop $\widetilde{\gamma}$ in X_2 starting at x_2 gives, as its terminal endpoint, a point x_3 with $p_{1*}(\pi_1(X_1, x_1)) = p_{2*}(\pi_1(X_2, x_3))$ (since it conjugates back!), and so, by the lifting criterion, there is an isomorphism $h : (X_1, x_1) \to (X_2, x_3)$. So conjugate subgroups give isomorphic coverings. Thus, for a path-connected, locally path-connected, semi-locally simply-connected space X, the image of the induced homomorphism on π_1 gives a one-to-one correspondence between [isomorphism classes of (connected) coverings of X] and [conjugacy classes of subgroups of $\pi_1(X)$].

So, for example, if you have a group G that you are interested in, you know of a (nice enough) space X with $\pi_1(X) \cong G$, and you know enough about the covering of X, then you can gain information about the subgroup structure of G. For example, and in some respects as motivation for all of this machinery!, a free group $F(\Sigma)$ is π_1 of a bouquet of circles X. Any covering space $\widetilde{X}$ of X is a union of vertices and edges, so is a graph. Collapsing a maximal tree to a point, $\widetilde{X}$ is $\sim$ a bouquet of circles, so has free π_1. So, every subgroup of a free group is free. (That is a lot shorter than the original, group-theoretic, proof...) A subgroup H of index n in $F(\Sigma)$ corresponds to a n-sheeted covering $\widetilde{X}$ of X. If $|\Sigma| = m$, then $\widetilde{X}$ will have n vertices and nm edges. Collapsing a maximal tree, having $n-1$ edges to a point, leaves a bouquet of $nm - n + 1$ circles, so $H \cong F(nm - n + 1)$. For example, for $m = 3$, index n subgroups are free on $2n + 1$ generators, so every free subgroup on 4 generators has infinite index in $F(3)$. Try proving that directly!

Every subgroup of a free group is free, because it is the fundamental group of a covering of a graph, i.e., of a graph. A subgroup H of index n in $F(\Sigma)$ corresponds to a n-sheeted covering $\widetilde{X}$ of X. If $|\Sigma| = m$, then $\widetilde{X}$ will have n vertices and nm edges. Collapsing a maximal tree, having $n - 1$ edges to a point, leaves a bouquet of $nm - n + 1$ circles, so $H \cong F(nm - n + 1)$. For example, for $m = 3$, index n subgroups are free on $2n + 1$ generators, so every free subgroup on 4 generators has infinite index in $F(3)$. Try proving that directly!

Note that for a graph Γ to be a covering of another graph, with k sheets, say, the number of vertices and edges of Γ must both be a mulitple of k. This little observation can be very useful when trying to decide what graphs Γ might cover!

Kurosh Subgroup Theorem: If $H < G_1 * G_2$ is a subgroup of a free product, then H is (isomorphic to) a free product of a collection of conjugates of subgroups of G_1 and G_2 and a free froup. The proof is to build a space by taking 2-complexes X_1 and X_2

with π_1's isomorphic to G_1, G_2 and join their basepoints by an arc. The covering space of this space X corresponding to H consists of spaces that cover X_1, X_2 (giving, after basepoint considerations, the conjugates) connected by a collection of arcs (which, suitably interpreted, gives the free group).

Residually finite groups: G is said to be residually finite if for every $g \neq 1$ there is a finite group F and a homomorphism $\varphi : G \to F$ with $\varphi(g) \neq 1$ in F. This amounts to saying that $g \notin$ the (normal) subgroup $\ker(\varphi)$, which amounts to saying that a loop corresponding to g does not lift to a loop in the finite-sheeted covering space corresponding to $\ker(\varphi)$. So residual finiteness of a group can be verified by building coverings of a space X with $\pi_1(X) = G$. For example, free groups can be shown to be residually finite in this way.

Ranks of free (sub)groups: A free group on n generators is isomorphic to a free group on m generators $\Leftrightarrow$ $n = m$; this is because the abelianizations of the two groups are $\mathbb{Z}^n, \mathbb{Z}^m$. The (minimal) number of generators for a free group is called its *rank*. Given a free group $G = F(a_1, \dots a_n)$ and a collection of words $w_1, \dots w_m \in G$, we can determine the rank and ndex of the subgroup it H they generate by building the corresponding cover. The idea is to start with a bouquet of m circles, each subdivided and labelled to spell out the words w_i. Then we repeatedly identify edges sharing on common vertex if they are labelled precisely the same (same letter *and* same orientation). This process is known as *folding*. One can inductively show that the (obvious) maps from these graphs to the bouquet of n circles X_n both have image H under the induced maps on π_1; since the map for the unfolded graph factors through the one for the folded graph, the image from the folded graph can only get smaller, but we can still spell out the same words as loops in the folded graph, so the image can also only have gotten bigger! We continue this folding process until there is no more folding to be done; the resulting graph X is what is known (in combinatorics) as a *graph covering*; the map to X_n is locally injective. If this map is a covering map, then our subgroup H has finite index (equal to the degree of the covering) and we can compute the rank of H (and a basis!) from the folded graph. If it is not a covering map, then the map is not locally surjective at some vertices; if we graft trees onto these vertices, we can extend the map to an (infinite-sheeted) covering map without changing the homotopy type of the graph. H therefore has infinite index in G, and its rank can be computed from $H \cong \pi_1(X)$.

Postscript: why care about covering spaces? The preceding discussion probably makes it clear that covering places play a central role in (combinatorial) group theory. It also plays a role in embedding problems; a common scenario is to have a map $f : Y \to X$ which is injective on π_1, and we wish to know if we can lift f to a finite-sheeted covering so that the lifted map $\widetilde{f}$ is homotopic to an embedding. Information that is easier to obtain in the case of an embedding can then be passed down to gain information abut the original map f. And covering spaces underlie the theory of analytic continuation in complex analysis; starting with a domain $D \subseteq \mathbb{C}$, what analytic continuation really builds is an (analytic) function from a covering space of D to $\mathbb{C}$. For example, the logarithm is really defined as a map from the universal cover of $\mathbb{C} \setminus \{0\}$ to $\mathbb{C}$. The various "branches" of the logarithm refer to which sheet in this cover you are in.