Math 872 Algebraic Topology

Problem Set # 6

Starred (*) problems due Thursday, March 29

(*) 27. Show that for chain maps f, g between chain complexes $\mathcal{A} = \{A_n\}, \mathcal{B} = \{B_n\}$, the relation "$f$ and g are chain homotopic" is an equivalence relation.

28. Show that if $A \subseteq X$, then the inclusion map $i : A \to X$ induces an isomorphism on homology groups $\Leftrightarrow H_n(X, A) = 0$ for all $n \geq 0$.

29. Show that if a short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *splits*, that is, there is a map $\gamma : B \to A$ with $\gamma \circ \alpha = I$, then the map $\varphi : B \to A \oplus C$ given by $b \mapsto (\gamma(b), \beta(b))$, is an isomorphism.

[This is part of the Splitting Lemma, proved in Hatcher, p.147. Splitting is equivalent to the existence of $\delta : C \to B$ satisfying $\beta \circ \delta = I$, but this is irrelevant to the question above.]

(*) 30. Show that if $A \subseteq X$ and $r : X \to A$ is a retraction, then for every n, $H_n(X) \cong H_n(A) \oplus H_n(X, A)$.

[Hint: show that the (piece of) the long exact homology sequence $H_n(A) \to H_n(X) \to H_n(X, A)$ is "really" $0 \to H_n(A) \to H_n(X) \to H_n(X, A) \to 0$, and splits.]

31. Prove the Snake Lemma: given a diagram of abelian groups

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & \\
0 & \longrightarrow & D & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 0
\end{array}$$

with the horizontal rows exact and where each rectangle commutes, then there are induced maps and a connecting homomorphism making the sequence

$$0 \to \ker\alpha \to \ker\beta \to \ker\gamma \to \mathrm{coker}\alpha \to \mathrm{coker}\beta \to \mathrm{coker}\gamma \to 0$$

exact. (For $f : R \to S$, $\mathrm{coker} f = S/\mathrm{im}(f)$.)

32. Compute the singular homology groups of the topologist's sine curve

$$X = \{(x, \sin(1/x) : 0 < x \leq 1\} \cup (\{0\} \times [-1, 1])$$