Math 872 Algebraic Topology

Homework # 1

Starred (*) problems due Thursday, January 25

1. Show that, in general, if $\alpha, \beta : I \to X$ are paths which are *homotopic rel endpoints*, $\alpha(0) = \beta(0) = x_0$, $\alpha(1) = \beta(1) = x_1$, then their associated change of basepoint maps are equal: $\widehat{\alpha} = \widehat{\beta} : \pi_1(X, x_0) \to \pi_1(X, x_1)$.

(*) 2. Show that for a path-connected space X, $\pi_1(X)$ is abelian $\Leftrightarrow$ the change of basepoint maps are all independent of path, i.e.,
$\alpha(0) = \beta(0) = x_0$, $\alpha(1) = \beta(1) = x_1 \Rightarrow \widehat{\alpha} = \widehat{\beta} : \pi_1(X, x_0) \to \pi_1(X, x_1)$.
[Notation: our hypothesis can be expressed symbolically by saying that α, β are maps of triples; $\alpha, \beta : (I, 0, 1) \to (X, x_0, x_1)$.]

(*) 3. Show that every homomorphism $\varphi : \mathbb{Z} \to \pi_1(x)$ can be realized as the induced homomorphism $\varphi = f_*$ of a continuous map $f : (S^1, 1) \to (X, x_0)$. [Hint: Look at $\varphi(1) \in \pi_1(X, x_0)$.]

4. If $x_0 \in A \subseteq X$ and A is path-connected, show that the inclusion-induced map $\iota_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is surjective $\Leftrightarrow$ every path $\gamma : I \to X$ with endpoints in A, $\gamma(0), \gamma(1) \in A$, is homotopic rel endpoints to a path in A, that is, $\gamma \simeq \alpha$ rel ∂I with $\alpha : I \to A \subseteq X$.

5. Show that the homomorphism $\varphi : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ given by $\varphi([\gamma]) = ((p_X)_*[\gamma], (p_Y)_*[\gamma])$ is an isomorphism, where p_X, p_Y are the projections of $X \times Y$ onto the first and second factors, respectively.