

### Math 856 Problem Set 3

Starred (\*) problems to be handed in Thursday, November 3

14. The tangent space for a manifold  $M$  with boundary is defined in exactly the same way as for a manifold; the derivations at a point in  $\partial M$  are allowed to point “in all the directions” of  $\mathbb{R}^n$ .

We say that a tangent vector  $X \in T_a M$  for  $a \in \partial M$  “points inward” if in some set of local coordinates  $h = (x^1, \dots, x^n)$  we have  $X = \sum_i v^i \partial/\partial x^i$  with  $v^n > 0$ . (Here  $h$  maps to the upper half-space, where  $x^n \geq 0$ .) Show that the notion of “pointing inward” is independent of coordinate chart.

15. [Lee, p.101, problem 4-9] Suppose that the map  $F : M \rightarrow N$  is a local diffeomorphism (that is, for every  $a \in M$ , there is a neighborhood  $\mathcal{U}$  of  $a$  so that  $F|_{\mathcal{U}} : \mathcal{U} \rightarrow F(\mathcal{U})$  is a diffeomorphism). Show that for every smooth vector field  $Y$  on  $N$  there is a unique smooth vector field  $X$  on  $M$  that is  $F$ -related to  $Y$ .

- (\*) 16. If  $X, Y$  are smooth tangent vector fields on  $M$ , and  $f, g \in C^\infty(M)$ , show that  $[fX, gY] = (fg)[X, Y] + (fXg)Y - (gYf)X$ . [Hint: evaluate on a third smooth function!]

- (\*) 17. [Lee, p.346, Problem 13-1] If  $M$  is a smooth manifold that is the union of two open subsets  $U, V$  with  $U \cap V$  connected, and if  $TM|_U$  and  $TM|_V$  are orientable bundles, show that  $M$  is orientable. Use this to show that  $S^n$  is orientable for every  $n \geq 1$ .

18. If  $\xi$  is a vector bundle, show that the Whitney sum  $\xi \oplus \xi$  is orientable.

- (\*) 19. [“Bundle Section Extension Lemma”] Given a smooth vector bundle  $p : E \rightarrow M$  over a smooth manifold  $M$ , a closed subset  $A \subseteq M$ , and a smooth section  $s : A \rightarrow E$  defined over  $A$  (that is, for every  $a \in A$  there is a neighborhood  $U_a$  of  $a$  in  $M$  and a smooth section  $s_U : U \rightarrow E$  so that  $s_u = s$  on  $A \cap U$ ), show that there is a global smooth section  $S : M \rightarrow E$  with  $S|_A = s$ . [Hint: partition of unity...]

20. [Lee, p.151, Problem 6-3] Show that the tangent bundle  $TM$  is trivial if and only if the cotangent bundle  $T^*M$  is also trivial.