Math 856 Homework 3
Starred (*) problems to be handed in Thursday, October 5

(*) 14: Show that if $M, N$ are smooth manifolds, $M$ is connected, and $f : M \to N$ is a smooth map with $f_\ast : T_aM \to T_{f(a)}N$ equal to the zero map for all $a \in M$, then $f$ is the constant function. (Hint: show that $f^{-1}\{f(a)\}$ is open! And beat the problem over the head with some calculus...)

(*) 15: For $a \in M$, let $\mathcal{F}_a \subseteq C^\infty(M)$ denote the smooth functions satisfying $f(a) = 0$.
and let $L : \mathcal{F}_a \to \mathbb{R}$ be a linear operator satisfying $L(fg) = 0$ for all $f, g \in \mathcal{F}_a$. Show that there is a unique derivation $X \in T_a M$ satisfying $X|_{\mathcal{F}_a} = L$.

(N.B. This provides still another characterization of tangent vectors, as the vector space of linear maps $X : \mathcal{F}_a/W \to \mathbb{R}$, where $W = \mathcal{F}_a^2 =$ the ideal generated by products $fg$ for $f, g \in \mathcal{F}_a$.)

16: The tangent space for a manifold $M$ with boundary is defined in exactly the same way as for a manifold; the derivations at a point in $\partial M$ are allowed to point “in all the directions” of $\mathbb{R}^n$.

We say that a tangent vector $X \in T_a M$ for $a \in \partial M$ “points inward” if in some set of local coordinates $h = (x^1, \ldots, x^n)$ we have $X = \sum_i v^i \partial/\partial x^i$ with $v^n > 0$. (Here $h$ maps to the upper half-space, where $x^n \geq 0$.) Show that the notion of “pointing inward” is independent of coordinate chart.

17: Show that the $C^\infty$ manifolds $T(M \times N)$ and $TM \times TN$ are diffeomorphic.

18: Show that if $M^n$ is a compact smooth manifold that admits a smooth embedding into $\mathbb{R}^{n+1}$, then $M^n \times S^k$ admits a smooth embedding into $\mathbb{R}^{n+k+1}$.

Show that for any $n_1, \ldots, n_k \geq 1$ and $N = \sum_i n_i$, $S^{n_1} \times \cdots \times S^{n_k}$ admits a smooth embedding into $\mathbb{R}^{N+1}$.