Math 856 Homework 1

Starred (*) problems to be handed in Thursday, September 7

(*) 1: Show that a connected manifold M is *arcwise connected*, that is, for every pair of points $x, y \in M$ there is a *one-to-one* path $\gamma : [0,1] \to M$ with $\gamma(0) = x, \gamma(1) = y$.

(There is a theorem, due to Hahn and Mazurkiewicz (circa 1914), which says that a Hausdorff space is path connected iff it is arcwise connected (which is kind of funny, since the term "Hausdorff" wasn't really introduced until the 1920s?); but for manifolds you can give a much more elementary proof...)

2: Show that if $A, B \subseteq \mathbb{R}^2$ are closed subsets, the statement

" $\mathbb{R}^2 \setminus A \cong \mathbb{R}^2 \setminus B \Rightarrow A \cong B$ "

is **false**. What about the converse statement? (N.B. That might be harder?)

(*) 3: Given a collection of triangles (or 2-simplices, you are more comfortable with that terminology) T_i, $i = 1, \dots 2r$, with edges e_{i1}, e_{i2}, e_{e3}, and a collection of $3r$ homeomorphisms $h_k : e_{i_k j_k} \to e_{i'_k j'_k}$ involving all $6r$ edges (as either domain or range), show (in a quasi-rigorous fashion?) that the quotient space obtained by gluing the 2-disks T_i together using the maps h_k is a 2-manifold. (There are basically three "kinds" of points to worry about. "Describe" locally Euclidean neighborhoods for each.)

(*) 4: (Lee, p. 28, problem 1-4) If $0 \leq k \leq \min\{m, n\}$, show that the set $R_k \subseteq M(m \times n, \mathbb{R})$ of m-by-n matrices with rank $\geq k$ is an open subset of $M(m \times n, \mathbb{R}) \cong \mathbb{R}^{mn}$ (and therefore admits a smooth structure). (*Hint:* look at Lee's linear algebra appendix...)

(*) 5.: We say that two charts $\phi : U \to \mathbb{R}^n$, $\psi : V \to \mathbb{R}^n$, $U, V \subseteq M^n$ are $\underline{C^\infty\text{-related}}$ if $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ and $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ are both C^∞. Show that the relation " is C^∞-related to " is **not** an equivalence relation. (Hint: $M^n = \mathbb{R}$ will suffice for an example...)

(*) 6: Show that $\mathbb{R}$ has uncountably many distinct smooth structures. ((Perhaps) show first that it is enough to find uncountably many charts, with intersecting domains and ranges, no two of which are C^∞-related to one another.)

7: Lee, page 28-29, problem 1-5. [It was too long to copy out.]

8: Show that a function $f : M^n \to N^m$ is $C^\infty \Leftrightarrow g \circ f : M^n \to \mathbb{R}$ is C^∞ for *every* C^∞ function $g : N^m \to \mathbb{R}$. (Hint: you might need to use the technology of bump functions found on p.55 of the text?)