Math 856 Homework 1

Starred (*) problems to be handed in Thursday, September 7

(*) 1: Show that a connected manifold $M$ is \emph{arcwise connected}, that is, for every pair of points $x, y \in M$ there is a one-to-one path $\gamma : [0, 1] \to M$ with $\gamma(0) = x; \gamma(1) = y$.

(There is a theorem, due to Hahn and Mazurkiewicz (circa 1914), which says that a Hausdorff space is path connected iff it is arcwise connected (which is kind of funny, since the term "Hausdorff" wasn't really introduced until the 1920s?); but for manifolds you can give a much more elementary proof...)

2: Show that if $A, B \subseteq \mathbb{R}^2$ are closed subsets, the statement

"$\mathbb{R}^2 \setminus A \cong \mathbb{R}^2 \setminus B \Rightarrow A \cong B$"

is \textbf{false}. What about the converse statement? (N.B. That might be harder?)

(*) 3: Given a collection of triangles (or 2-simplices, you are more comfortable with that terminology) $T_i, i = 1, \ldots, 2r$, with edges $e_{i1}, e_{i2}, e_{i3}$, and a collection of $3r$ homeomorphisms $h_k : e_{ik} \to e'_{i'k}$ involving all $6r$ edges (as either domain or range), show (in a quasi-rigorous fashion?) that the quotient space obtained by gluing the 2-disks $T_i$ together using the maps $h_k$ is a 2-manifold. (There are basically three "kinds" of points to worry about. "Describe" locally Euclidean neighborhoods for each.)

(*) 4: (Lee, p. 28, problem 1-4) If $0 \leq k \leq \min\{m, n\}$, show that the set $R_k \subseteq M(m \times n, \mathbb{R})$ of $m$-by-$n$ matrices with rank $\geq k$ is an open subset of $M(m \times n, \mathbb{R}) \cong \mathbb{R}^{mn}$ (and therefore admits a smooth structure). (Hint: look at Lee's linear algebra appendix...)

(*) 5: We say that two charts $\phi : U \to \mathbb{R}^n$, $\psi : V \to \mathbb{R}^n$, $U, V \subseteq M^n$ are $\mathcal{C}^\infty$-related if $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ and $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ are both $\mathcal{C}^\infty$. Show that the relation "is $\mathcal{C}^\infty$-related to " is \textbf{not} an equivalence relation. (Hint: $M^n = \mathbb{R}$ will suffice for an example...)

(*) 6: Show that $\mathbb{R}$ has uncountably many distinct smooth structures. ((Perhaps) show first that it is enough to find uncountably many charts, with intersecting domains and ranges, no two of which are $\mathcal{C}^\infty$-related to one another.)

7: Lee, page 28-29, problem 1-5. [It was too long to copy out.]

8: Show that a function $f : M^n \to N^m$ is $\mathcal{C}^\infty$ \iff $g \circ f : M^n \to \mathbb{R}$ is $\mathcal{C}^\infty$ for \emph{every} $\mathcal{C}^\infty$ function $g : N^m \to \mathbb{R}$. (Hint: you might need to use the technology of bump functions found on p.55 of the text?)