Math 445 Number Theory

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Number theory is about *finding* and *explaining* patterns in numbers.

Ulam Sprial:

36	35	34	33	32	31	30
37	16	15	14	13	12	29
38	17	4	3	2	11	28
39	18	5	0	1	10	27
40	19	6	7	8	9	26
41	20	21	22	23	24	25
42	43	44	45	46	47	48

Place the natural numbers in a rectangular spiral. The primes tend to fall on certain diagonal lines with more frequency than it seems they should?

This means: for certain values of α, β, γ, the sequences $n^2 + \alpha$, $n^2 + n + \beta$, $n^2 - n + \gamma$ have more primes than we *expect* them to.

Why? We don't yet know...

Egyptian fractions:
Any rational number m/n can be written as a sum of reciprocals $1/a$ of integers. In fact, by repeatedly subtracting the largest reciprocal that we can from whatever is left, we find that
$$\frac{m}{n} = \frac{1}{a_1} + \cdots + \frac{1}{a_k}$$
with $a_1 < a_2 < \ldots < a_k$ and $k \leq n$. But not every fraction $3/n$ can be expressed as a sum of *two* reciprocals (e.g, 3/7). However, it is conjectured (the Erdös-Strauss Conjecture) that
$$\text{every fraction } \frac{4}{n} \text{ is the sum of at most 3 reciprocals.}$$
This has been verified to $n = 10^{14}$, but still remains open.

These sorts of expressions actually occur in engineering: If resistors with resistances $r_1, \ldots, r_n$ are wired in parallel, they act as a single resistor with resistance r, where
$$\frac{1}{r} = \frac{1}{r_1} + \cdots + \frac{1}{r_n}$$
so a resistor with 'custom' resistance r can be 'manufactured' from a set of standard resistors by solving such equations.

Perfect numbers: A number N is perfect if it is equal to the sum of its proper divisors. Euler showed that every even perfect number must be of the form $N = 2^n(2^{n+1} - 1)$ where $p = 2^{n+1} - 1$ is prime. Such primes are known as *Mersenne* primes; there are only 44 currently known Mersenne primes, including the 4 (or so?) largest known primes. It is still an open question whether or not there exists an odd perfect number.

If the divisors of N are $1 = a_0, a_1, \ldots, a_n = N$ in order, then to be perfect we need $N = a_0 + \cdots + a_{n-1}$. Dividing by N yields
$$1 = \frac{1}{a_n} + \cdots + \frac{1}{a_1}$$
and so techinques from Egyptian fractions can be employed. For example, it is known that the largest denomenator, $a_n = N$ must be $\leq u_n$ where the u_i are a fixed sequence defined by $u_1 = 1$ and $u_{i+1} = u_i(u_i + 1)$. These provide an upper bound on the size of an odd perfect number with exactly k factors. Techniques such as this and others have enabled researchers to show that any odd perfect number (if one exists) has at least 300 digits, at least 9 distinct prime factors, and at least 57 prime factors in all......