Math 445 Homework 9

Due Monday, December 6

40. Find the rational solutions to the equation $x^2 + 3y^2 = 7$

41. Show that the equation $13x^2 + 31y^2 - 71z^2 = 0$ has no non-trivial integer solutions.

42. Show that the equation $19x^2 + 31y^2 - 71z^2 = 0$ has non-trivial integer solutions.

43. [NZM, p.240, # 5.4.11] Show that for any prime modulus p, the equation

$$(x^2 - 17)(x^2 - 19)(x^2 - 323) \equiv 0 \pmod{p}$$

always has a solution.

(N.B.: The result is also true for *any* modulus n; one makes use of Hensel's Lemma (p. 87 of NZM) to show it.)

44. (NZM, p.260, # 5.6.4) Let $f(x, y)$ be a polynomial with real coefficients and degree d, and set

$$p(t) = (1 + t^2)^d f\left(\frac{2t}{1 + t^2}, \frac{1 - t^2}{1 + t^2}\right)$$

[Note: $\left(\frac{2t}{1 + t^2}, \frac{1 - t^2}{1 + t^2}\right)$ is a popular parametrization of the unit circle].

Show that $p(t)$ is a polynomial, with degree at most $2d$. Conclude that if $\mathcal{C}_f(\mathbb{R})$ meets the unit circle $\{(x, y) : x^2 + y^2 = 1\}$ in more than $2d$ points, then it *contains* the unit circle.