

Math 445 Homework 8 Solutions

36. Let h_n/k_n (as usual) denote the n^{th} convergent of the continued fraction expansion of the irrational number x . Show by example that it is possible for $b < k_{n+1}$ and $\left|x - \frac{a}{b}\right| < \left|x - \frac{h_n}{k_n}\right|$.

Most any irrational number works. E.g., $\sqrt{5} = [2, 4, 4, \dots] = 2.236067977\dots$, has convergents $\frac{2}{1} = 2$ and $\frac{9}{4} = 2.25$, but $\frac{7}{3} = 2.3333\dots$ has $|\sqrt{5} - \frac{7}{3}| < |\sqrt{5} - 2|$ and $3 < 4$.

37. Show that for any $c > 2$, there are only finitely many pairs of integers a, b with $|\sqrt{2} - \frac{a}{b}| < \frac{1}{b^c}$.

For any $c > 2$, $c - 2 > 0$, and so there is an integer $b_0 > 0$ with $b_0^{c-2} > 2$. Then $b \geq b_0$ implies $b^{c-2} \geq b_0^{c-2} > 2$. Then if such a b would work, $|\sqrt{2} - \frac{a}{b}| < \frac{1}{b^c} = \frac{1}{b^{c-2}} \cdot \frac{1}{b^2} < \frac{1}{2b^2}$, we have, from class, that $\frac{a}{b} = \frac{h_n}{k_n}$ for some n . So only finitely many denominators *other than convergents* will work. For each of these denominators, the fractions a/b are all $1/b$ apart, so at most 2 can be within $1/b$ of $\sqrt{2}$, so at most 2 are within $1/b^c < 1/b$. So only finitely many a/b are not convergents.

To finish, we also need to show that only finitely many can be convergents. But we know that for any convergent $r_n = \frac{h_n}{k_n}$, r_{n+2} is closer to $\sqrt{2}$, and on the same side of $\sqrt{2}$, as r_n . So $|\sqrt{2} - r_n| > |r_{n+2} - r_n| = \left| \frac{(-1)^n a_n}{k_{n+2} k_n} \right| = \frac{2}{k_{n+2} k_n}$ for $n \geq 2$ (by a result from class). But $k_{n+2} = 2k_{n+1} + k_n = 2(2k_n + k_{n-1}) + k_n = 5k_n + 2k_{n-1} < 7k_n$, since $k_{n-1} < k_n$. So $|\sqrt{2} - r_n| > \frac{2}{k_{n+2} k_n} > \frac{2}{7k_n^2}$. But for any $c > 2$, $\frac{2}{7k_n^2} < \frac{1}{k_n^c}$ for only finitely many n ; we need $k_n^{c-2} < 7/2$, which is true only for $k_n < (7/2)^{1/(c-2)}$. So only finitely many k_n will work, with, as before, at most 2 numerators for each. So only finitely many rational numbers will meet the stated bound.

38. Let p be prime and suppose $u^2 \equiv -1 \pmod{p}$ (so $p \equiv 1 \pmod{4}$). Let $[a_0, \dots, a_n]$ be the continued fraction expansion of $\frac{u}{p}$, and let i be the largest integer with $k_i \leq \sqrt{p}$. Show

that $\left|\frac{h_i}{k_i} - \frac{u}{p}\right| < \frac{1}{k_i \sqrt{p}}$, and $|h_i p - k_i u| < \sqrt{p}$. Setting $x = k_i$ and $y = h_i p - u k_i$, show that $p|x^2 + y^2|$ and $x^2 + y^2 < 2p$, so $x^2 + y^2 = p$.

We know that $\left|\frac{h_i}{k_i} - \frac{u}{p}\right| < \left|\frac{h_i}{k_i} - \frac{h_{i+1}}{k_{i+1}}\right| = \frac{1}{k_i k_{i+1}} < \frac{1}{k_i \sqrt{p}}$, by the choice of i . So, $|h_i p - k_i u| = \left|\frac{h_i}{k_i} - \frac{u}{p}\right|(k_i p) < \frac{1}{k_i \sqrt{p}}(k_i p) = \sqrt{p}$. If we set $x = k_i \geq 1$ and $y = h_i p - u k_i$, then $x^2 + y^2 = k_i^2 + (h_i p - u k_i)^2 < (\sqrt{p})^2 + (\sqrt{p})^2 = p + p = 2p$. And since $u^2 = pN - 1$ for some

$N, x^2 + y^2 = k_i^2 + (h_i p - u k_i)^2 = k_i^2(1 + u^2) + p(ph_i^2 - 2uh_i k_i) = p(k_i^2 N + ph_i^2 - 2uh_i k_i) \equiv 0 \pmod{p}$. So $0 < x^2 + y^2 < 2p$ and is a multiple of p , so $x^2 + y^2 = p$, as desired.

39. Show that for n a positive integer that is not a perfect square (translation: the continued fraction expansion of \sqrt{n} never terminates), that at every stage of the continued fraction expansion of $x = \sqrt{n}$

$$x = [a_0, a_1, \dots, a_{k-1}, a_k + x_k]$$

x_k is always of the form $x_k = \frac{\sqrt{n} - c}{b}$, where $c, b \in \mathbb{N}$ and $b|n - c^2$. Conclude that the continued fraction expansion of \sqrt{n} will eventually repeat, with a period of length at most $n \lfloor \sqrt{n} \rfloor$.

$\sqrt{n} = [\lfloor \sqrt{n} \rfloor + (\sqrt{n} - \lfloor \sqrt{n} \rfloor)]$, so $x_0 = \sqrt{n} - \lfloor \sqrt{n} \rfloor = \frac{\sqrt{n} - \lfloor \sqrt{n} \rfloor}{1}$ and $1|(n - (\lfloor \sqrt{n} \rfloor)^2)$, as desired.

Continuing by induction, if we assume that $x_k = \frac{\sqrt{n} - c_k}{b_k}$, where $c_k, b_k \in \mathbb{N}$ and $b_k|n - c_k^2$, then writing $n - c_k^2 = b_k d_k = (\sqrt{n} - c_k)(\sqrt{n} + c_k)$, we have $a_{k+1} = \lfloor \frac{b_k}{\sqrt{n} - c_k} \rfloor = \lfloor \frac{\sqrt{n} + c_k}{d_k} \rfloor = N$ for some N , and then $x_{k+1} = \frac{\sqrt{n} + c_k}{d_k} - N = \frac{\sqrt{n} - (Nd_k - c_k)}{d_k} = \frac{\sqrt{n} - c_{k+1}}{b_{k+1}}$, where $c_{k+1} = Nd_k - c_k$ and $b_{k+1} = d_k$. To finish, we need to show that $b_{k+1}|n - c_{k+1}^2$, but $n - c_{k+1}^2 = n - (Nd_k - c_k)^2 = (n - c_k^2) + d_k(2Nc_k - N^2 d_k) = b_k d_k + d_k(2Nc_k - N^2 d_k) = d_k(b_k + 2Nc_k - N^2 d_k) = d_k M = b_{k+1} M$, as desired.

So by induction, for every $k \geq 0$, $x_k = \frac{\sqrt{n} - c}{b}$, where $c, b \in \mathbb{N}$ and $b|n - c^2$.