Math 445 Homework 8

Due Wednesday, November 10

36. Let h_n/k_n (as usual) denote the n^{th} convergent of the continued fraction expansion of the irrational number x. Show by example that it need **not** be true that

$$\left|x-\frac{a}{b}\right| < \left|x-\frac{h_n}{k_n}\right| \text{ implies } b \geq k_{n+1}$$

37. [NZM, p.344, # 7.6.3] Show that for any $c > 2$, there are only finitely many pairs of integers a, b with $|\sqrt{2}-\frac{a}{b}| < \frac{1}{b^c}$.

38. [NZM, p. 333, # 7.3.6] Let p be prime and suppose $u^2 \equiv -1 \pmod{p}$ (so $p \equiv 1 \pmod{4}$). Let $[a_0, \ldots, a_n]$ be the continued fraction expansion of $\frac{u}{p}$, and let i be the largest integer with $k_i \leq \sqrt{p}$. Show that $|\frac{h_i}{k_i} - \frac{u}{p}| < \frac{1}{k_i\sqrt{p}}$, and $|h_i p - k_i u| < \sqrt{p}$. Setting $x = k_i$ and $y = h_i p - u k_i$, show that $p | x^2 + y^2$ and $x^2 + y^2 < 2p$, so $x^2 + y^2 = p$.

39. Show that for n a positive integer that is not a perfect square (translation: the continued fraction expansion of $\sqrt{n}$ never terminates), that at every stage of the continued fraction expansion of $x = \sqrt{n}$

$$x = [a_0, a_1, \ldots, a_{k-1}, a_k + x_k]$$

x_k is always of the form $x_k = \frac{\sqrt{n}-a}{b}$, where $a, b \in \mathbb{N}$ and $b | n - a^2$. Conclude that the continued fraction expansion of $\sqrt{n}$ will eventually repeat, with a period of length at most $n\lfloor\sqrt{n}\rfloor$.

Hint: by induction! In the inductive step, write $\frac{b}{\sqrt{n}-a} = \frac{\sqrt{n}+a}{c}$, and then find the fractional part of this. For the second half, how long must you wait before the x_k's *must* repeat themselves?