Math 445 Homework 7 Solutions

31. Continued fraction expansions:

$53/18$: $53 = 18 \cdot 2 + 17$, $18 = 17 \cdot 1 + 1$, and $17 = 1 \cdot 17 + 0$, so
$$\frac{53}{18} = 2 + \frac{17}{18} = 2 + \frac{1}{\frac{18}{17}} = 2 + \cfrac{1}{1 + \cfrac{1}{17}} = [2; 1, 17]$$

$115/53$: $115 = 53 \cdot 2 + 9$, $53 = 9 \cdot 5 + 8$, $9 = 8 \cdot 1 + 1$, $8 = 1 \cdot 8 + 0$, so $\frac{115}{53} = [2; 5, 1, 8]$

32. If $x = [a_0, \ldots, a_n, b]$ and $y = [a_0, \ldots, a_n, c]$ with $b < c$, then $x < y$ if n is odd, and $x > y$ is n is even.

By induction; for $n = 0$, $x = [a_0, b] = a_0 + \frac{1}{b} > a_0 + \frac{1}{c} = [a_o, c] = y$, since $b < c \Rightarrow \frac{1}{b} > \frac{1}{c}$.

Inductively, if we assume that whenever $B < C$, $[a_0, \ldots, a_{n-1}, B] - [a_0, \ldots, a_{n-1}, C] = (-1)^{n-1}P$, where $P > 0$, then

$x = [a_0, \ldots, a_n, b] = [a_0, \ldots, a_{n-1}, a_n + \frac{1}{b}]$ and $y = [a_0, \ldots, a_n, c] = [a_0, \ldots, a_{n-1}, a_n + \frac{1}{c}]$ have $B = a_n + \frac{1}{c} < a_n + \frac{1}{b} = C$, so $[a_0, \ldots, a_n, c] - [a_0, \ldots, a_n, b] = [a_0, \ldots, a_{n-1}, B] - [a_0, \ldots, a_{n-1}, C] = (-1)^{n-1}P$, so $[a_0, \ldots, a_n, b] - [a_0, \ldots, a_n, c] = (-1)^n P$, as desired.

So by induction, $x < y$ if n is odd, and $x > y$ is n is even. Or, without induction:

We know that, for $[a_0, \ldots, a_{n-1}] = \frac{h_{n-1}}{k_{n-1}}$ and $[a_0, \ldots, a_n] = \frac{h_n}{k_n}$ that $x = \frac{h_n b + h_{n-1}}{k_n b + k_{n-1}}$ and $y = \frac{h_n c + h_{n-1}}{k_n c + k_{n-1}}$. If we look at $x - y \frac{h_n b + h_{n-1}}{k_n b + k_{n-1}} - \frac{h_n c + h_{n-1}}{k_n c + k_{n-1}} = \frac{(h_n b + h_{n-1})(k_n c + k_{n-1}) - (h_n c + h_{n-1})(k_n b + k_{n-1})}{(k_n b + k_{n-1})(k_n c + k_{n-1})}$, since the denominator is positive, this will be positive ($x > y$) or negative ($x < y$) depending on the sign of the numerator. But

$(h_n b + h_{n-1})(k_n c + k_{n-1}) - (h_n c + h_{n-1})(k_n b + k_{n-1}) = (h_n k_n bc + h_{n-1} k_n c + h_n k_{n-1} b + h_{n-1} k_{n-1}) - (h_n k_n bc + h_{n-1} k_n b + h_n k_{n-1} b + h_{n-1} k_{n-1}) = (h_{n-1} k_n - h_n k_{n-1})(c - b) = (-1)^n (c - b)$, which, since $c - b > 0$, is positive when n is even, and negative when n is odd.

33. The continued fraction expansion of $\sqrt{17}$;

$a_0 = \lfloor \sqrt{17} \rfloor = 4, r_0 = \sqrt{17} - 4$, $a_1 = \lfloor \frac{1}{\sqrt{17} - 4} \rfloor = \lfloor \sqrt{17} + 4 \rfloor = 8$, $r_1 = (\sqrt{17} + 4) - 8 = \sqrt{17} - 4 = r_0$, and then the process will repeat,

so $\sqrt{17} = [4, 8, 8, 8, 8, \ldots] = [4, \overline{8}]$.

Using our formulas $h_i = h_{i-1}a_i + h_{i-2}$, $k_i = k_{i-1}a_i + k_{i-2}$, we have
$\frac{h_0}{k_0} = \frac{4}{1}, \frac{h_1}{k_1} = \frac{4 \cdot 8 + 1}{1 \cdot 8 + 0} = \frac{33}{8}, \frac{h_2}{k_2} = \frac{33 \cdot 8 + 4}{8 \cdot 8 + 1} = \frac{268}{65}, \frac{h_3}{k_3} = \frac{268 \cdot 8 + 33}{65 \cdot 8 + 8} = \frac{2177}{528}, \frac{h_4}{k_4} = \frac{2177 \cdot 8 + 268}{528 \cdot 8 + 65} = \frac{17684}{4289}$.

34. The continued fraction expansion of $\sqrt{19}$: $4 < \sqrt{19} < 5$. so:
$a_0 = \lfloor\sqrt{19}\rfloor = 4, r_0 = \sqrt{19} - 4$, $a_1 = \lfloor\frac{1}{\sqrt{19}-4}\rfloor = \lfloor\frac{\sqrt{19}+4}{3}\rfloor = 2$, $r_1 = \frac{\sqrt{19}+4}{3} - 2 = \frac{\sqrt{19}-2}{3}$, $a_2 = \lfloor\frac{3}{\sqrt{19}-2}\rfloor = \lfloor\frac{\sqrt{19}+2}{5}\rfloor = 1$, $r_2 = \frac{\sqrt{19}+2}{5} - 1 = \frac{\sqrt{19}-3}{5}$, $a_2 = \lfloor\frac{5}{\sqrt{19}-3}\rfloor = \lfloor\frac{\sqrt{19}+3}{2}\rfloor = 3$, $r_2 = \frac{\sqrt{19}+3}{2} - 3 = \frac{\sqrt{19}-3}{2}$, $a_3 = \lfloor\frac{2}{\sqrt{19}-3}\rfloor = \lfloor\frac{\sqrt{19}+3}{5}\rfloor = 1$, $r_3 = \frac{\sqrt{19}+3}{5} - 1 = \frac{\sqrt{19}-2}{5}$, $a_4 = \lfloor\frac{5}{\sqrt{19}-2}\rfloor = \lfloor\frac{\sqrt{19}+2}{3}\rfloor = 2$, $r_4 = \frac{\sqrt{19}+2}{3} - 2 = \frac{\sqrt{19}-4}{3}$, $a_5 = \lfloor\frac{3}{\sqrt{19}-4}\rfloor = \lfloor\frac{\sqrt{19}+4}{1}\rfloor = 8$, $r_5 = \frac{\sqrt{19}+4}{8} - 16 = \frac{\sqrt{19}-4}{1} = r_0$, and then the process will repeat,
so $\sqrt{19} = [4, 2, 1, 3, 1, 2, 8, 2, 1, 3, 1, 2, 8, \ldots] = [4, \overline{2, 1, 3, 1, 2, 8}]$.
Using our formulas $h_i = h_{i-1}a_i + h_{i-2}$, $k_i = k_{i-1}a_i + k_{i-2}$, we have
$\frac{h_0}{k_0} = \frac{4}{1}, \frac{h_1}{k_1} = \frac{4 \cdot 2 + 1}{1 \cdot 2 + 0} = \frac{9}{2}, \frac{h_2}{k_2} = \frac{9 \cdot 1 + 4}{2 \cdot 1 + 1} = \frac{13}{3}, \frac{h_3}{k_3} = \frac{13 \cdot 3 + 9}{3 \cdot 3 + 2} = \frac{48}{11}, \frac{h_4}{k_4} = \frac{48 \cdot 1 + 13}{11 \cdot 1 + 3} = \frac{61}{14}$.

35. If $\alpha < \beta < \gamma$ are irrational numbers, $\alpha = [a_0, a_1 \ldots]$, $\beta = [b_0, b_1, \ldots]$, $\gamma = [c_0, c_1, \ldots ...]$, and $a_i = c_i$ for $0 \le i \le n$, then $a_i = b_i = c_i$ for $0 \le i \le n$.

By induction: for $n = 0$, we have $a_0 = \lfloor\alpha\rfloor \le \lfloor\beta\rfloor \le \lfloor\gamma\rfloor = a_0$, so $\lfloor\beta\rfloor = a_0$.
If we assume that $a_i = b_i = c_i$ for $0 \le i \le k < n$, then
$\alpha = [a_0, \ldots, a_k, a_{k+1}+x_{k+1}], \beta = [a_0, \ldots, a_k, b_{k+1}+y_{k+1}]$, and $\gamma = [a_0, \ldots, a_k, a_{k+1}+z_{k+1}]$ with $0 < x_{k+1}, y_{k+1}, y_{k+1} < 1$. But since $\alpha < \beta < \gamma$, we claim that by Problem # 32, $a_{k+1} + x_{k+1} < b_{k+1} + y_{k+1} < c_{k+1} + z_{k+1}$ (if $k + 1$ is odd; the opposite inequalities if $k + 1$ is even). This is because we can't have any equalities; the resulting continued fractions would then be equal, contradicting $\alpha < \beta < \gamma$. And the inequalities cannot run the other way, since then Problam #32 and the parity of k would say that one of the inequalities $\alpha < \beta < \gamma$ would have to run the other way, a contradiction. But then
$a_{k+1} = \lfloor a_{k+1} + x_{k+1}\rfloor \le b_{k+1} = \lfloor b_{k+1} + y_{k+1}\rfloor \le \lfloor c_{k+1} + z_{k+1}\rfloor = c_{k+1} = a_{k+1}$, so $a_{k+1} = b_{k+1}$, as desired.

So, by induction, $a_k = b_k = c_k$ for all $0 \le k \le n$.