31. Continued fraction expansions:

\[
\frac{53}{18} : 53 = 18 \cdot 2 + 17 , 18 = 17 \cdot 1 + 1 , \text{ and } 17 = 1 \cdot 17 + 0 , \text{ so } \frac{53}{18} = 2 + \frac{1}{18 + \frac{1}{17}} = [2; 1, 17]
\]

\[
\frac{115}{53} : 115 = 53 \cdot 2 + 9 , 53 = 9 \cdot 5 + 8 , 9 = 8 \cdot 1 + 1 , 8 = 1 \cdot 8 + 0 , \text{ so } \frac{115}{53} = [2; 5, 1, 8]
\]

32. If \(x = [a_0, \ldots, a_n, b]\) and \(y = [a_0, \ldots, a_n, c]\) with \(b < c\), then \(x < y\) if \(n\) is odd, and \(x > y\) is \(n\) is even.

By induction; for \(n = 0\), \(x = [a_0, b] = a_0 + \frac{1}{b} > a_0 + \frac{1}{c} = [a_0, c] = y\), since \(b < c \Rightarrow \frac{1}{b} > \frac{1}{c}\).

Inductively, if we assume that whenever \(B < C, [a_0, \ldots, a_n, B] - [a_0, \ldots, a_n, C] = (-1)^{n-1}P\), where \(P > 0\), then

\[
x = [a_0, \ldots, a_n, b] = [a_0, \ldots, a_n-1, a_n + \frac{1}{b}] \quad \text{and} \quad y = [a_0, \ldots, a_n, c] = [a_0, \ldots, a_n-1, a_n + \frac{1}{c}]
\]

\([a_0, \ldots, a_n-1, a_n + \frac{1}{c}]\) have \(B = a_n + \frac{1}{c} < a_n + \frac{1}{b} = C\), so

\([a_0, \ldots, a_n, c] - [a_0, \ldots, a_n, b] = [a_0, \ldots, a_n-1, B] - [a_0, \ldots, a_n-1, C] = (-1)^{n-1}P\), so \([a_0, \ldots, a_n, b] - [a_0, \ldots, a_n, c] = (-1)^n P\), as desired.

So by induction, \(x < y\) if \(n\) is odd, and \(x > y\) is \(n\) is even. Or, without induction:

We know that, for \([a_0, \ldots, a_n-1]\) = \(\frac{h_{n-1}}{k_n}\) and \([a_0, \ldots, a_n]\) = \(\frac{h_n}{k_n}\) that \(x = \frac{h_n b + h_{n-1}}{k_n b + k_{n-1}}\) and \(y = \frac{h_n c + h_{n-1}}{k_n c + k_{n-1}}\). If we look at \(x - y = \frac{h_n b + h_{n-1} - h_n c - h_{n-1}}{k_n b + k_{n-1} - k_n c - k_{n-1}}\), since the denominator is positive, this will be positive (\(x > y\)) or negative (\(x < y\)) depending on the sign of the numerator. But

\((h_n b + h_{n-1})(k_n c + k_{n-1}) - (h_n c + h_{n-1})(k_n b + k_{n-1}) = (h_n k_n b c + h_{n-1} k_n c + h_n k_n b + h_{n-1} k_n c) - (h_n k_n b c + h_{n-1} k_n b + h_n k_n b + h_{n-1} k_n c) = (h_n - h_{n-1})k_n - h_n k_{n-1})(c - b) = (-1)^n(c - b)\), which, since \(c - b > 0\), is positive when \(n\) is even, and negative when \(n\) is odd.

33. The continued fraction expansion of \(\sqrt{17}\):

\(a_0 = [\sqrt{17}] = 4, r_0 = \sqrt{17} - 4\), \(a_1 = [\frac{1}{\sqrt{17} - 4}] = [\sqrt{17} + 4] = 8\),

\(r_1 = (\sqrt{17} + 4) - 8 = \sqrt{17} - 4 = r_0\), and then the process will repeat,

so \(\sqrt{17} = [4, 8, 8, 8, 8, \ldots] = [4, \overline{8}]\).
Using our formulas \( h_i = h_{i-1}a_i + h_{i-2} \), \( k_i = k_{i-1}a_i + k_{i-2} \), we have

\[
\begin{align*}
0 & = 4 \quad 1 \\
1 & = 4 \cdot 8 + 1 \quad 8 \\
2 & = 33 \quad 8 \cdot 8 + 1 \\
3 & = 33 \cdot 8 + 4 \quad 8 \cdot 8 + 1 \\
4 & = 268 \quad 8 \cdot 8 + 1 \\
5 & = 268 \cdot 8 + 33
\end{align*}
\]

34. The continued fraction expansion of \( \sqrt{19} \): \( 4 < \sqrt{19} < 5 \). so:

\[
\begin{align*}
a_0 &= \lfloor \sqrt{19} \rfloor = 4, r_0 = \sqrt{19} - 4, \quad a_1 = \lfloor \frac{1}{\sqrt{19} - 4} \rfloor = \lfloor \frac{1}{3} \rfloor = 2, \\
r_1 &= \sqrt{19} + 4 - 2 = \frac{\sqrt{19} + 2}{3}, \quad a_2 = \lfloor \frac{3}{\sqrt{19} - 2} \rfloor = \lfloor \frac{3}{5} \rfloor = 1, \quad a_3 = \lfloor \frac{5}{\sqrt{19} - 3} \rfloor = \lfloor \frac{5}{2} \rfloor = 2, \\
r_2 &= \sqrt{19} + 2 - 1 = \frac{\sqrt{19} - 3}{5}, \quad a_4 = \lfloor \frac{2}{\sqrt{19} - 3} \rfloor = \lfloor \frac{2}{5} \rfloor = 1, \quad r_3 = \sqrt{19} + 3 - 2 = \frac{\sqrt{19} - 1}{3}, \quad a_5 = \lfloor \frac{5}{\sqrt{19} - 2} \rfloor = \lfloor \frac{5}{3} \rfloor = 1, \\
r_4 &= \sqrt{19} + 3 - 2 = \frac{1}{3} = \frac{19}{3}, \quad a_6 = \lfloor \frac{3}{\sqrt{19} - 4} \rfloor = \lfloor \frac{3}{1} \rfloor = 3, \quad r_5 = \sqrt{19} + 4 - 16 = \frac{19}{4} = r_0, \quad \text{and then the process will repeat},
\end{align*}
\]

so \( \sqrt{19} = [4, 2, 1, 3, 1, 2, 8, 2, 1, 3, 1, 2, 8, \ldots] = [4, 2, 1, 3, 1, 2, 8] \).

Using our formulas \( h_i = h_{i-1}a_i + h_{i-2} \), \( k_i = k_{i-1}a_i + k_{i-2} \), we have

\[
\begin{align*}
0 & = 4 \quad 1 \\
1 & = 4 \cdot 2 + 1 \quad 2 \\
2 & = 9 \quad 2 \cdot 1 + 1 \\
3 & = 9 \cdot 1 + 4 \quad 2 \cdot 1 + 1 \\
4 & = 13 \quad 3 \cdot 3 + 2 \\
5 & = 13 \cdot 3 + 9 \quad 3 \cdot 3 + 2
\end{align*}
\]

35. If \( \alpha < \beta < \gamma \) are irrational numbers, \( \alpha = [a_0, a_1, \ldots] \), \( \beta = [b_0, b_1, \ldots] \), \( \gamma = [c_0, c_1, \ldots] \), and \( a_i = c_i \) for \( 0 \leq i \leq n \), then \( a_i = b_i = c_i \) for \( 0 \leq i \leq n \).

By induction: for \( n = 0 \), we have \( a_0 = [\alpha] \leq [\beta] \leq [\gamma] = a_0 \), so \( [\beta] = a_0 \).

If we assume that \( a_i = b_i = c_i \) for \( 0 \leq i < k < n \), then

\[
\alpha = [a_0, a_k, a_k + z_{k+1}], \quad \beta = [a_0, a_k, b_k + y_{k+1}], \quad \text{and} \quad \gamma = [a_0, a_k, a_k + z_{k+1}]
\]

with \( 0 < x_{k+1}, y_{k+1}, y_{k+1} < 1 \). But since \( \alpha < \beta < \gamma \), we claim that by Problem # 32, \( a_k + x_{k+1} < b_k + y_{k+1} < c_k + z_{k+1} \) (if \( k + 1 \) is odd; the opposite inequalities if \( k + 1 \) is even). This is because we can’t have any equalities; the resulting continued fractions would then be equal, contradicting \( \alpha < \beta < \gamma \). And the inequalities cannot run the other way, since then Problem # 32 and the parity of \( k \) would say that one of the inequalities \( \alpha < \beta < \gamma \) would have to run the other way, a contradiction. But then

\[
\alpha_k = [a_k, a_k + x_{k+1}] \leq b_{k+1} = [b_k, y_{k+1}] \leq [c_k, z_{k+1}] = c_k + 1 = a_k + 1\)

so \( a_k + 1 = b_{k+1} \), as desired.

So, by induction, \( a_k = b_k = c_k \) for all \( 0 \leq k \leq n \).