Math 445 Number Theory

Introduction to/Review of concepts from abstract algebra

An integer p is *prime* if whenever $p = ab$ with $a, b \in \mathbb{Z}$, either $a = \pm n$ or $b = \pm n$.
[For sanity's sake, we will take the position that primes should <u>also</u> be ≥ 2 .]

Fundamental Theorem of Arithmetic: Every integer $n \geq 2$ can be expressed as a product of primes; $n = p_1 \cdot \cdots \cdot p_k$.

If we insist that the primes are written in increasing order, $p_1 \leq \ldots \leq p_k$, then this representation is *unique.*

The Division Algorithm: For any integers $n \geq 0$ and $m > 0$, there are *unique* integers q and r with $n = mq + r$ and $0 \leq r \leq m - 1$.

[Note: this is also true for any integers n, m with $m \neq 0$, although you need to replace "$m - 1$" with "$|m - 1|$" .]

The basic idea: keep repeatedly subtracting m from n until what's left is less than m.

Notation: $b|a$ = "b divides a" = "b is a divisor of a" = "a is a multiple of b", means $a = bk$ for some integer k .

If $b|a$ and $a \neq 0$, then $|b| \leq |a|$.

If $a|b$ and $b|c$, then $a|c$

If $a|c$ and $b|d$, then $ab|cd$

If p is prime and $p|ab$, then either $p|a$ or $p|b$

Notation: $(a, b) = \gcd(a, b)$ = greatest common divisor of a and b

Different, equivalent, formulations for $d = (a, b)$:

(1) $d|a$ and $d|b$, and if $c|a$ and $c|b$, then $c \leq d$.

(2) d is the smallest *positive* number that can be written as $d = ax + by$ with $a, b \in \mathbb{Z}$.

(3) $d|a$ and $d|b$, and if $c|a$ and $c|b$, then $c|d$.

(4) d is the *only* positive divisor of a and b that can be expressed as $d = ax + by$ with $a, b \in \mathbb{Z}$.

If $c|a$ and $c|b$, then $c|(a, b)$

If $c|ab$ and $(c, a) = 1$, then $c|b$

If $a|c$ and $b|c$, and $(a, b) = 1$, then $ab|c$

If $a = bq + r$, then $(a, b) = (b, r)$

Euclidean Algorithm: This last fact gives us a way to compute (a,b), using the division algorithm:

Starting with $a > b$, compute $a = bq_1 + r_1$, so $(a, b) = (b, r_1)$. Then compute $b = r_1 q_2 + r_2$, and repeat: $r_{i-1} = r_i q_{i+1} + r_{i+1}$. Continue until $r_{n+1} = 0$, then $(a, b) = (b, r_1) = (r_1, r_2) = \ldots = (r_n, r_{n+1}) = (r_n, 0) = r_n$.

Since $b > r_1 > r_2 > r_3 > \ldots$, this process must end, by well-orderedness.

We can reverse these calculations to recover $(a, b) = ax + by$, by rewriting each equation in our algorithm as $r_{i+1} = r_{i-1} - r_i q_{i+1}$, and then repeatedly substituting the higher equations into the lowest one, in turn, working up through the list of equations.

Congruence modulo n : Notation: $a \equiv b \pmod{n}$ (also written $a \underset{n}{\equiv} b$) means $n|(b-a)$

Equivalently: the division algorithm will give the same remainder for a and b when you divide by n

Congruence mod n is an *equivalence relation*

The *congruence class* of a mod n is the collection of all integers congruent mod n to a:

$[a]_n = \{b \in \mathbb{Z} : a \underset{n}{\equiv} b\} = \{b \in \mathbb{Z} : n|(b-a)\}$

Fermat's Little Theorem. If p is prime and $(a,p)=1$, then $a^{p-1} \underset{p}{\equiv} 1$

Because: $(a\cdot 1)(a\cdot 2)(a\cdot 3)\cdots(a\cdot(p-1)) \underset{p}{\equiv} 1\cdot 2\cdot 3\cdots(p-1)$, and $(1\cdot 2\cdot 3\cdots(p-1),p)=1$. Same idea, looking at the a's between 1 and $n-1$ that are relatively prime to n (and letting $\phi(n)$ be the number of them), gives

If $(a,n)=1$, then $a^{\phi(n)} \underset{n}{\equiv} 1$.

If the prime factorization of n is $p_1^{\alpha_1}\cdots p_k^{\alpha_k}$, then $\phi(n) = [p_1^{\alpha_1-1}(p_1-1)]\cdots[p_k^{\alpha_k-1}(p_k-1)]$

The integers $\mathbb{Z}$, the integers mod n $\mathbb{Z}_n$, the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$ are all *rings*.

A *homomorphism* is a function $\varphi : R \to S$ from a ring R to a ring S satisfying:
for any $r, r' \in R$, $\varphi(r+r') = \varphi(r)+\varphi(r')$ and $\varphi(r\cdot r') = \varphi(r)\cdot\varphi(r')$.

The basic idea is that it is a function that "behaves well" with respect to addition and multiplication.

An *isomorphism* is a homomorphism that is both one-to-one and onto. If there is an isomorphism from R to S, we say that R and S are *isomorphic*, and write $R \cong S$.

Example: if $(m,n)=1$, then $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$. The isomorphism is given by
$\varphi([x]_{mn}) = ([x]_m, [x]_n)$

The main ingredients in the proof:

If $\varphi : R \to S$ and $\psi : R \to T$ are ring homomorphisms, then the function $\omega : R \to S \times T$ given by $\omega(r) = (\varphi(r), \psi(r))$ is also a homomorphism.

If $m|n$, then the function $\varphi : \mathbb{Z}_n \to \mathbb{Z}_m$ given by $\varphi([x]_n) = [x]_m$ is a homomorphism.

Together, these give that the function we want above is a homomorphism. The fact that $(m,n)=1$ implies that φ is one-to-one; then the Pigeonhole Principle implies that it is also onto!

The above isomorphism and induction imply that if $n_1,\ldots n_k$ are *pairwise relatively prime* (i.e., if $i \neq j$ then $(n_i,n_j)=1$), then

$\mathbb{Z}_{n_1\cdots n_k} \cong \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$. This implies:

The Chinese Remainder Theorem: If $n_1,\ldots n_k$ are pairwise relatively prime, then for any $a_1,\ldots a_k \in \mathbb{N}$ the system of equations

$x \equiv a_i \pmod{n_i}$, $i = 1,\ldots k$

has a solution, and any two solutions are congruent modulo $n_1\cdots n_k$.

A solution can be found by (inductively) replacing a pair of equations $x \equiv a \pmod{n}$, $x \equiv b$ (mod m), with a single equation $x \equiv c \pmod{nm}$, by solving the equation $a+nk = x = b+mj$ for k and j, using the Euclidean Algorithm.