Name:

Math 445 Old Final Exam

Show all work. How you get your answer is just as important, if not more important, than the answer itself. If you think it, write it!

These problems are provided for illustrative purposes only; they do not necessarilty reflect the scope or structure of your actual final exam!

1. Show that for any integers $k \geq 1$ and $a \geq 2$,
$$k | \phi(a^k - 1)$$
(Hint: What can you say about the order of a mod $m = a^k - 1$?)

2. Show that if $(a, 15) = (b, 15) = 1$, then *either* $15 | a^4 - b^4$ *or* $15 | a^4 + b^4$.

3. Find the **period of** the repeating decimal expansion of $\dfrac{1}{47}$.

4. Use continued fractions to find a solution to the Diophantine equation
$$x^2 - 43y^2 = -2$$

5. Show that if $(p, q) = 1$ and the modular equations
$$x^2 + y^2 \equiv 3 \pmod{p} \quad \text{and} \quad u^2 + v^2 \equiv 3 \pmod{q}$$
have solutions, then the equation
$$r^2 + s^2 \equiv 3 \pmod{pq}$$
has a solution.
(Hint: By adding multiples of p and q (respectively), show that you can arrange solutions with $x = u, y = v$.)

6. Show that if $n \equiv 3 \pmod{4}$, then the Diophantine equation
$$x^2 - ny^2 = -1$$
has no solution.

7. Find the sum of the points $A = (1, 5)$ and $B = (3, 7)$ on the elliptic curve defined by the function
$$f(x, y) = y^2 - (x^3 - x + 25)$$
where $\underline{0}$ is chosen to be $\underline{0} = (0, -5)$.