

Rational points on curves:

The geometric process we have developed works for any equation $ax^2 + by^2 + cz^2 = 0$, i.e., $aX^2 + bY^2 + c = 0$, for which we know a single rational solution (X_0, Y_0) , to find all rational solutions to $aX^2 + bY^2 + c = 0$ (and hence all integer solutions to $ax^2 + by^2 + cz^2 = 0$). Looking at lines with rational slope through our known solution, we can find its points of intersection with the ellipse/hyperbola $aX^2 + bY^2 + c = 0$ by finding the roots of a quadratic equation

$$aX^2 + b(r(X - X_0) + y_0)^2 + c = 0$$

with rational coefficients, for which we already know one root, X_0 . The other root (which depends on the variable r) is then also rational.

For example, knowing that $2x^2 + 3y^2 = 5$ has solution $(x_0, y_0) = (1, 1)$ we find, using the line $y = r(x - 1) + 1$ with rational slope r through $(1, 1)$, that

$$0 = 2x^2 + 3y^2 - 5 = 2x^2 + 3(rx - r + 1)^2 - 5 = 2x^2 + 3(r^2x^2 - 2r^2x + 2rx + r^2 - 2r + 1) - 5 = (2 + 3r^2)x^2 + (6r - 6r^2)x + (3r^2 - 6r - 2) = ((x - 1)((2 + 3r^2)x - (3r^2 - 6r - 2))), \text{ so } x = 1 \text{ or } x = \frac{3r^2 - 6r - 2}{3r^2 + 2}, \text{ giving } y = rx - r + 1 = r\frac{3r^2 - 6r - 2}{3r^2 + 2} - r + 1 = -\frac{3r^2 + 4r - 2}{3r^2 + 2}. \text{ Setting } r = \frac{u}{v}, \text{ we get, as before,}$$

$$2(3u^2 - 6uv - 2v^2)^2 + 3(3u^2 + 4uv - 2v^2)^2 = 5(3u^2 + 2v^2)^2. \text{ For example, setting } u = 11, v = 4, \text{ we have } 2(67)^2 + 3(507)^2 = 5(395)^2.$$

Or, starting from $2^2 + 5^2 = 29$, we find, after solving

$$x^2 + (r(x - 2) + 5)^2 - 29 = 0 = (r^2 + 1)x^2 + (10r - 4r^2)x + (4r^2 - 20r - 4) = (x - 2)((r^2 + 1) - (2r^2 - 10r - 2)),$$

that $x = \frac{2r^2 - 10r - 2}{r^2 + 1}$ and $y = r\left(\frac{2r^2 - 10r - 2}{r^2 + 1} - 2\right) + 5 = -\frac{5r^2 + 4r - 5}{r^2 + 1}$. So setting $r = \frac{u}{v}$, we get $(2u^2 - 10uv - 2v^2)^2 + (5u^2 + 4uv - 5v^2)^2 = 29(u^2 + v^2)^2$. This gives (after dividing these three terms by their common g.c.d., and then multiplying by a common factor) all integer solutions to $x^2 + y^2 = 29z^2$. For example, setting $u = 11, v = 28$, we have $(1754)^2 + (4547)^2 = 29(905)^2$.

The hard part: finding the first solution! For the special situation $x^2 + y^2 = nz^2$, we know from a homework problem awhile back that if $X^2 + Y^2 = n$ has a rational solution, then it has an *integer* solution, which we can look for by an exhaustive search among $0 \leq X, Y \leq \sqrt{n}$! Note that our newest result gives us a quick criterion to decide if $x^2 + y^2 = n$ has an integer solution, when n is square-free; we need $\left(\frac{-1}{n}\right) = 1$. The interested reader can check that this really is equivalent to the (slightly more long-winded) answer we found before...