What can $h_s^2 - nk_s^2$ equal?

Wherever we choose to stop the continued fraction expansion of $\sqrt{n} = [\lfloor \sqrt{n} \rfloor, a_1, \ldots, a_{m-1}, \lfloor \sqrt{n} \rfloor]$, we find that

$$\sqrt{n} = \frac{\sqrt{n} + m_s}{q_{s+1}},$$

and

$$\sqrt{n} = \frac{\sqrt{n} + m_s}{q_{s+1}} = \frac{(\sqrt{n} + m_s)h_s + q_{s+1}h_{s-1}}{(\sqrt{n} + m_s)k_s + q_{s+1}k_{s-1}}.$$ 

Then

$$\sqrt{n}(m_s k_s + q_{s+1}k_{s-1} - h_s) = (m_s k_s + q_{s+1}h_{s-1} - nk_s)$$

so both sides of this equation are 0 (otherwise $\sqrt{n}$ is rational!), so $h_s = m_s k_s + q_{s+1}k_{s-1}$ and $nk_s = m_s k_s + q_{s+1}h_{s-1}$. Then

$$h_s^2 - nk_s^2 = h_s(m_s k_s + q_{s+1}k_{s-1}) - k_s(m_s k_s + q_{s+1}h_{s-1}) = q_{s+1}(h_s k_{s-1} - h_{s-1} k_s) = (-1)^{s-1}q_{s+1}.$$

So the only $N$ with $|N| \leq \sqrt{n}$ for which $x^2 - ny^2 = N$ can be solved are (squares and) those for which $N = (-1)^{s-1}q_{s+1}$ where $\zeta_{s+1} = (\sqrt{n} + m_s)/q_{s+1}$.

Focusing on $N = 1$, note that since $\zeta_0 = \frac{\sqrt{n} + \lfloor \sqrt{n} \rfloor}{1}$, $m_0 = \lfloor \sqrt{n} \rfloor$ and $q_1 = 1$. Then since $\zeta_0 = \zeta_m = \zeta_{2m} = \cdots$, we have

$q_{mt+1} = 1$ for all $t \geq 0$. So $h^2_{m-1} - nk^2_{m-1} = (-1)^m$.

If $m$ is even, then we have found a solution to $x^2 - ny^2 = 1$. If $m$ is odd, then apply the same reasoning, except with two periods of the continued fraction: $\sqrt{n} = [a_0, \ldots, a_{m-1}, a_m, \ldots, a_{2m-1}, \lfloor \sqrt{n} + a_0 \rfloor]$, and the same argument shows that $h^2_{2m-1} - nk^2_{2m-1} = (-1)^{2m} = 1$. In general, taking $t$ periods, we get $h^2_{tm-1} - nk^2_{tm-1} = (-1)^{tm}$. So we have shown that $x^2 - ny^2 = 1$ always has a solution; $x = h_{2m-1}, y = k_{2m-1}$ where $m$ is the period of the continued fraction of $\sqrt{n}$, will always work.

This is best illustrated with an example! Taking $n = 19$, we have

$$a_0 = 4, \ x_0 = \sqrt{19} - 4, \ \zeta_1 = \frac{\sqrt{19} + 4}{3}, \ \ a_1 = 2, \ x_1 = \frac{\sqrt{19} - 2}{3}, \ \zeta_2 = \frac{\sqrt{19} + 2}{5},$$

$$a_2 = 1, \ x_2 = \frac{\sqrt{19} - 3}{5}, \ \zeta_3 = \frac{\sqrt{19} + 3}{2}, \ \ a_3 = 3, \ x_3 = \frac{\sqrt{19} - 3}{2}, \ \zeta_4 = \frac{\sqrt{19} + 3}{5},$$

$$a_4 = 1, \ x_4 = \frac{\sqrt{19} - 2}{5}, \ \zeta_5 = \frac{\sqrt{19} + 2}{3}, \ \ a_5 = 2, \ x_5 = \frac{\sqrt{19} - 4}{3}, \ \zeta_6 = \frac{\sqrt{19} + 4}{1},$$

$$a_6 = 8, \ x_6 = \sqrt{19} - 4 = x_0,$$

and we can compute

$$h_0 = 4, h_1 = 9, h_2 = 13, h_3 = 48, h_4 = 61, h_5 = 170, h_6 = 1421, \ldots$$

$$k_0 = 1, k_1 = 2, k_2 = 3, k_3 = 11, k_4 = 14, k_5 = 39, k_6 = 325, \ldots$$

From our analysis above, $(h_5)^2 - 19(k_5)^2 = (-1)^6 = 1$, so $(170, 39)$ is a solution to $x^2 - 19y^2 = 1$. Also, the values of $(-1)^{s-1}q_{s+1}$ are $-3, 5, -2, 5, -3, 1, -3, 5, -2, 5, \ldots$, so among $-4, -3, \ldots, 3, 4$, the only $N$ for which $x^2 - 19y^2 = N$ has a solution are $N = -3, -2,$ and 1 (and 4, because it is a perfect square). By continuing to compute convergents, we can find infinitely many solutions for each such equation.