From last time: If \( x \not\in \mathbb{Q} \) and \( b \in \mathbb{Z} \) with \( 1 \leq b < k_{n+1} \), then for any \( a \in \mathbb{Z} \), \( |bx - a| \geq |k_n x - h_n| \).

Another sense in which convergents are the best possible rational approximations:

If \( x \not\in \mathbb{Q} \) and \( a, b \in \mathbb{Z} \) have \( |x - \frac{a}{b}| < \frac{1}{2b^2} \), then \( \frac{a}{b} = \frac{h_n}{k_n} \) for some \( n \).

The idea: if not, then \( |ak_n - bh_n| \geq 1 \) for every \( n \). But since \( k_n \to \infty \) as \( n \to \infty \), there is an \( n \) with \( k_n \leq b < k_{n+1} \). Then from above we know that \( |xk_n - h_n| \leq |xb - a| = |x - \frac{a}{b}| \cdot |b| < \frac{1}{2b^2} |b| = \frac{1}{2b} \). So \( |x - \frac{h_n}{k_n}| < \frac{1}{2bk_n} \), and then

\[
\frac{1}{bk_n} \leq \frac{|bh_n - ak_n|}{bk_n} = |\frac{a}{b} - \frac{h_n}{k_n}| = |(\frac{a}{b} - x) + (x - \frac{h_n}{k_n})| \leq |\frac{a}{b} - x| + |x - \frac{h_n}{k_n}| < \frac{1}{2b^2} + \frac{1}{2bk_n}.
\]

So \( \frac{1}{bk_n} = \frac{1}{bk_n} - \frac{1}{2bk_n} \leq \frac{1}{2b^2} \), so \( 2b^2 < 2bk_n \), so \( b < k_n \), a contradiction. So \( \frac{a}{b} = \frac{h_n}{k_n} \) for some \( n \).

**Pell’s Equation:** solve \( x^2 - ny^2 = N \) with \( x, y \in \mathbb{Z} \). (WOLOG, \( x, y \geq 0 \))

If \( n < 0 \), then \( N = x^2 - ny^2 \geq x^2 + y^2 \Rightarrow x, y \leq \sqrt{N} \); can check all cases.

If \( n = m^2 \) is a perfect square, then \( N = x^2 - ny^2 = (x - my)(x + my) \Rightarrow x - my = a, x + my = b \) with \( ab = N \), and so \( 2x = a + b, 2my = b - a \). Again, we can just check all factorizations \( ab = N \) to see what works.

If \( n > 0 \) is not a perfect square, then we can use the continued fraction expansion of \( \sqrt{n} \) to shed light on the solutions. If \( N > 0 \), then \( N = x^2 - ny^2 = (x - \sqrt{ny})(x + \sqrt{ny}) \), so \( 0 < \frac{N}{x + \sqrt{ny}} = x - \sqrt{ny} \), so \( \frac{|N|}{x + \sqrt{ny}} \cdot |y| = |\sqrt{n} - \frac{x}{y}| \).

And since \( x - \sqrt{ny} > 0, x > \sqrt{ny} \), so \( \frac{x}{\sqrt{ny}} > 1 \) so \( \frac{x}{\sqrt{ny}} + 1 = \frac{x + \sqrt{ny}}{\sqrt{ny}} > 2 \), so \( x + \sqrt{ny} > 2\sqrt{ny} \) so

\[
\frac{|\sqrt{n} - \frac{x}{y}|}{|x + \sqrt{ny}| \cdot |y|} < \frac{|N|}{2ny} \cdot |y| = \frac{|N|}{2ny} \cdot \frac{1}{2y^2}.
\]

So if \( 0 < N < \sqrt{n} \), then \( x^2 - ny^2 = N \Rightarrow |\sqrt{n} - \frac{x}{y}| < \frac{1}{2y^2} \Rightarrow \frac{x}{y} \) is a convergent of \( \sqrt{n} \).

(A similar argument works for \( -\sqrt{n} < N < 0 \).)

Which makes it more interesting to understand the convergents of \( \sqrt{n} \) ! The basic idea: \( x \) has a repeating continued fraction expansion \( x = [a_0, \ldots, a_n, b_0, \ldots, b_m] \Leftrightarrow x = r + s\sqrt{t} \) for some \( r, s \in \mathbb{Q}, t \in \mathbb{Z} \).

To see this, set \( \alpha = [b_0, \ldots, b_m] \), so \( x = [a_0, \ldots, a_n, \alpha] \). If \( [a_0, \ldots, a_n] = \frac{h_n}{k_n} \), then \( x = [a_0, \ldots, a_n, \alpha] = \frac{h_n\alpha + h_{n-1}}{k_n\alpha + k_{n-1}}. \) If \( \alpha = r_0 + \sqrt{t}, \) then \( x = \frac{h_n(r_0 + s_0\sqrt{t}) + h_{n-1}}{k_n(r_0 + s_0\sqrt{t}) + k_{n-1}} = \frac{(h_n(s_0)(\sqrt{t}) + (h_n r_0 + h_{n-1})}{(k_n(s_0)(\sqrt{t}) + (k_n r_0 + h_{n-1})} = \frac{((h_n s_0)(\sqrt{t}) + (h_n r_0 + h_{n-1}))(k_n s_0)(\sqrt{t}) - (k_n r_0 + h_{n-1})}{k_n s_0^2 t - (k_n r_0 + h_{n-1})^2} \sqrt{t} = r + s\sqrt{t} \) with \( r, s \in \mathbb{Q} \).