Math 445 Number Theory

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Continued fractions: or, what happens when we "re-interpret" the Euclidean algorithm.

To compute (a,b), we write $a = bq_1 + r_1$, $b = r_1q_2 + r_2$, and repeat; $r_i = r_{i+1}q_{i+2} + r_{i+2}$, until $r_n = 0$. Then $r_{n-1} = (a,b)$. But if $(a,b) = 1$ (so the last equation is $r_{n-2} = 1 \cdot q_n + 0$) and we rewrite these calculations as

$$\frac{a}{b} = q_1 + \frac{r_1}{b} \ , \ \frac{b}{r_1} = q_2 + \frac{r_2}{r_1} \ , \ \dots \ , \ \frac{r_i}{r_{i+1}} = q_{i+2} + \frac{r_{i+2}}{r_{i+1}} \ , \ \frac{r_{n-2}}{1} = q_n + 0$$

then we can use them to express $\frac{a}{b}$ as a *continued fraction*:

$$\frac{a}{b} = q_1 + \frac{r_1}{b} = q_1 + \frac{1}{\frac{b}{r_1}} = q_1 + \frac{1}{q_2 + \frac{r_2}{r_1}} = \cdots = q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \cfrac{1}{\cdots + \cfrac{1}{q_n}}}}$$

For simplicity of notation, we will denote this continued fraction as $\langle q_1, q_2, \dots, q_n \rangle$ or $[q_1, q_2, \dots, q_n]$, depending on whether or not we want to use the same notation as the book, or as everybody else on the planet. These continued fraction expansions are called *simple*, because the numerators are all 1, and the denomenators are all positive integers. A more general theory need not require this.

And there is no reason to limit this to rational numbers! If we use the Euclidean algorithm to "compute" the gcd of $x \in \mathbb{R}$ and 1, we would compute

$x = 1 \cdot a_0 + r_0$, i.e., $a_0 = \lfloor x \rfloor$, $r_0 = x - a_0 = x - \lfloor x \rfloor$

$1 = r_0 a_1 + r_1$, i.e., $\frac{1}{r_0} = a_1 + \frac{r_1}{r_0}$ with $a_1 \in \mathbb{N}$ and $r_1 < r_0$, i.e., $a_1 = \lfloor \frac{1}{r_0} \rfloor$, $r_1 = \frac{1}{r_0} - \lfloor \frac{1}{r_0} \rfloor$

and, in general, $a_i = \lfloor \frac{1}{r_{i-1}} \rfloor$, $r_i = \frac{1}{r_{i-1}} - \lfloor \frac{1}{r_{i-1}} \rfloor$

and we write $x = [a_0, a_1, \dots, a_{n-1}, a_n + r_n] = [a_0, a_1, \dots, a_{n-1}, a_n + \dots]$. For irrational numbers x, the process will not terminate. The finite continued fractions $x_n = [a_0, a_1, \dots, a_{n-1}, a_n]$ are called the *convergents* of x.

For example, if we apply this to $x = \sqrt{13}$, we find

$a_0 = \lfloor \sqrt{13} \rfloor = 3, r_0 = \sqrt{13} - 3$, $a_1 = \lfloor \frac{1}{\sqrt{13}-3} \rfloor = \lfloor \frac{\sqrt{13}+3}{4} \rfloor = 1$, $r_1 = \frac{\sqrt{13}+3}{4} - 1 = \frac{\sqrt{13}-1}{4}$, $a_2 = \lfloor \frac{4}{\sqrt{13}-1} \rfloor = \lfloor \frac{\sqrt{13}+1}{3} \rfloor = 1$, $r_2 = \frac{\sqrt{13}+1}{3} - 1 = \frac{\sqrt{13}-2}{3}$, $a_2 = \lfloor \frac{3}{\sqrt{13}-2} \rfloor = \lfloor \frac{\sqrt{13}+2}{3} \rfloor = 1$, $r_2 = \frac{\sqrt{13}+2}{3} - 1 = \frac{\sqrt{13}-1}{3}$, $a_3 = \lfloor \frac{3}{\sqrt{13}-1} \rfloor = \lfloor \frac{\sqrt{13}+1}{4} \rfloor = 1$, $r_3 = \frac{\sqrt{13}+1}{4} - 1 = \frac{\sqrt{13}-3}{4}$, $a_4 = \lfloor \frac{4}{\sqrt{13}-3} \rfloor = \lfloor \frac{\sqrt{13}+3}{1} \rfloor = 6$, $r_4 = \frac{\sqrt{13}+3}{1} - 6 = \frac{\sqrt{13}-3}{1} = r_0$,

and then the process will repeat. So, $\sqrt{13} = [3,1,1,1,1,6,1,1,1,1,6,\dots] = [3,\overline{1,1,1,1,6}]$.