Math 445 Number Theory

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Recap: we know that the *Legendre symbol*, for p an odd prime and $(a,p)=1$, satisfies $\left(\frac{a}{p}\right)=(-1)^n$, where $n=|A|=$ the number of elements in A, where $A=\{k : a_k>\frac{p}{2}\}$, where $ak=pt_k+a_k$ with $0\leq a_k\leq p-1$. We have also seen that if a is *odd* and $(a,p)=1$, then $\left(\frac{a}{p}\right)=(-1)^t$, where $t=\sum_{j=1}^{\frac{p-1}{2}}\lfloor\frac{aj}{p}\rfloor$. Along the way we learned that

$$(a-1)\sum_{j=1}^{\frac{p-1}{2}} j=p(t-n)+2\sum_{i=1}^{n} q_i \quad\text{and}\quad \sum_{j=1}^{\frac{p-1}{2}} j=\frac{1}{2}(\frac{p-1}{2})(\frac{p-1}{2}+1)=\frac{p^2-1}{8}$$

When $a=2$, this last equation tells us that, mod 2, $\frac{p^2-1}{8}\equiv p(t-n)\equiv(t-n)$. But in this case $t=0$, since each of $\lfloor\frac{aj}{p}\rfloor=\lfloor\frac{2j}{p}\rfloor=0$, since $2j<p$ for $1\leq j\leq\frac{p-1}{2}$. So $\frac{p^2-1}{8}\equiv -n\equiv n \pmod 2$, so

$$\left(\frac{2}{p}\right)=(-1)^n=(-1)^{\frac{p^2-1}{8}}$$

Finally, we have the means to prove *Gauss' Law of Quadratic Reciprocity:*

Theorem: If p and q are distinct odd primes, then $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right)=(-1)^{(\frac{p-1}{2})(\frac{q-1}{2})}$.

This is because $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right)=(-1)^{t_1}(-1)^{t_2}=(-1)^{t_1+t_2}$, where

$$t_1=\sum_{i=1}^{\frac{p-1}{2}}\lfloor\frac{qi}{p}\rfloor \text{ and } t_2=\sum_{j=1}^{\frac{q-1}{2}}\lfloor\frac{pj}{q}\rfloor .$$

But for every pair (i,j), with $1\leq i\leq\frac{p-1}{2}$ and $1\leq j\leq\frac{q-1}{2}$, exactly one of $qi>pj$ or $qi<pj$ is true. So $S_1=\{(i,j) : qi>pj\}$ and $S_2=\{(i,j) : qi<pj\}$ are disjoint sets whose union is the set of all pairs. So $|S_1|+|S_2|=(\frac{p-1}{2})(\frac{q-1}{2})$. But for each fixed i, the j's with $(i,j)\in S_1$ are those which satisfy $j<\frac{qi}{p}$, so there are $\lfloor\frac{qi}{p}\rfloor$ of them, so S_1 has $\sum_{i=1}^{\frac{p-1}{2}}\lfloor\frac{qi}{p}\rfloor=t_1$ elements. Similarly, for each fixed j, the i's with $(i,j)\in S_2$ are those which satisfy $i<\frac{pj}{q}$, so there are $\lfloor\frac{pj}{q}\rfloor$ of them, so S_2 has $\sum_{i=1}^{\frac{q-1}{2}}\lfloor\frac{pj}{q}\rfloor=t_2$ elements. Consequently, $t_1+t_2=|S_1|+|S_2|=(\frac{p-1}{2})(\frac{q-1}{2})$, as desired.

These facts,
$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right)=(-1)^{(\frac{p-1}{2})(\frac{q-1}{2})}$ if p,q distinct odd primes, $\left(\frac{2}{p}\right)=(-1)^n=(-1)^{\frac{p^2-1}{8}}$, and $\left(\frac{-1}{p}\right)=(-1)^n=(-1)^{\frac{p-1}{2}}$
allow us to carry out the calculations of Legendre symbols much more simply than Euler's criterion would! For example

$\left(\frac{17}{31}\right)\left(\frac{31}{17}\right)=(-1)^{(\frac{17-1}{2})(\frac{31-1}{2})}=(-1)^{8\cdot 15}=1$, so $\left(\frac{17}{31}\right)=\left(\frac{31}{17}\right)$. But $\left(\frac{31}{17}\right)=\left(\frac{2\cdot 17-3}{17}\right)=\left(\frac{-3}{17}\right)=\left(\frac{-1}{17}\right)\left(\frac{3}{17}\right)=(-1)^8\left(\frac{3}{17}\right)=\left(\frac{3}{17}\right)$, while $\left(\frac{3}{17}\right)\left(\frac{17}{3}\right)=(-1)^{8\cdot 1}=1$, so $\left(\frac{3}{17}\right)=\left(\frac{17}{3}\right)=\left(\frac{3\cdot 6-1}{3}\right)=\left(\frac{-1}{3}\right)=(-1)^1=-1$, so $\left(\frac{17}{31}\right)=-1$, and so the equation $x^2\equiv 17 \pmod{31}$ has no solutions.