Note that

\[ \left| x - \frac{\ln k_n}{k_n} \right| < \frac{1}{k_n} \] is not better than we deserve.

For a randomly chosen \( x \) and a random \( N \), we can expect to find an \( M \) so that \( \left| x - \frac{M}{N} \right| < \frac{1}{2N} \), but we can't expect to get much closer than that.

The point, of course, is that the \( k_n \)'s are not randomly chosen! They are "determined" by \( x \).

\[
\star =
\]

There is also a precise sense in which the \( \frac{\ln k_n}{k_n} \)'s are the best approx's to \( x \):

If \( x \in \mathbb{Q} \) and \( a, b \in \mathbb{Z} \) with \( b > 0 \) and

\[
\left| x - \frac{a}{b} \right| < \frac{1}{2b^2},
\]

then \( a = \frac{\ln k_n}{k_n} \) for some \( n \).

If \( \left| x - \frac{a}{b} \right| \geq \frac{1}{2b^2} \) for all \( n \).

\[ \star \]

**Proof.** Suppose not. Then since \( k_n \to \infty \) as \( n \to \infty \) and \( k_0 = 1 \), \( \exists n \) with \( k_n \leq b < k_{n+1} \).

Then from the above we know that

\[
\left| x k_n - h_n \right| \leq \left| x b - a \right| < \left| x - \frac{\ln k_n}{k_n} \right| < \frac{1}{2b}
\]

So \( \left| x - \frac{\ln k_n}{k_n} \right| < \frac{1}{2b k_n} \) then
\[
\frac{1}{b'k_n} \leq \frac{b'h_n - a'k_n}{b'k_n} = \left| \frac{a}{b'} - \frac{b_n}{k_n} \right|
\]

\[
= \left| \left( \frac{a}{b'} - x \right) + \left( x - \frac{b_n}{k_n} \right) \right| \leq \left| x - \frac{a}{b'} \right| + \left| x - \frac{b_n}{k_n} \right|
\]

\[
\leq \frac{1}{2b'^2} + \frac{1}{2b'k_n}
\]

\[
\frac{1}{2b'k_n} = \frac{1}{b'k_n} - \frac{1}{2b'k_n} \leq \frac{1}{2b'^2} \Rightarrow 2b'^2 \geq 2b'k_n \Rightarrow b'k_n \neq \ast.
\]

Did: Pell's Eqn.

\( n < 0, \quad n = p^2 \) cases.

Stated main case, \( N < n^2 \)

Then \( x^2 - ny^2 = N \Rightarrow \frac{x}{y} = \frac{b_n}{k_n} \) some \( n \).
If \( x = \langle a_0, \ldots, a_n, k_0, \ldots, k_m \rangle \), then

\( x \) is a quadratic irrational.

(ie, any root of \( ax^2 + bx + c = 0 \) where \( a, b, c \in \mathbb{R} \)).

If \( x = \langle a_0, \ldots, k_m \rangle \) so that

\[ x = \langle a_0, \ldots, a_n, \alpha \rangle \quad \Rightarrow \]

\[ x = \frac{\alpha k_n + k_{n-1}}{\alpha k_n + k_{n-1}} \]

for \( \langle a_0, \ldots, a_n \rangle = \frac{b_n}{k_n} \). \( \alpha \) = \frac{\alpha k_n + k_{n-1}}{\alpha k_n + k_{n-1}}.

If \( \alpha = \langle x \rangle = \langle a_0, \ldots, k_m \rangle \), then

\[ x = \frac{(p + q \alpha)k_n + k_{n-1}}{k_n(q \alpha^2 - (pk_n + k_{n-1}))} \]

\[ = \frac{(p + q \alpha)k_n + k_{n-1}}{(k_nq \alpha^2 - (pk_n + k_{n-1}))} \]

\[ = (\alpha k_n + (q \alpha^2)) \sqrt{r} \]

But \( \alpha = \langle k_0, \ldots, k_m \rangle = \langle k_0, \ldots, k_m, \alpha \rangle \)

\[ \Rightarrow \alpha = \frac{\alpha k_n + k_{n-1}}{\alpha k_n + k_{n-1}} \]

\[ k_n \alpha^2 + (k_{n-1} - k_m) \alpha - k_{n-1} = 0 \]

\[ \Rightarrow \alpha = p + q \sqrt{r} \] for some \( p, q \).
\[ x = \frac{p \sqrt{a}}{q} \quad \text{x is conjugate} = \frac{p}{q} \]

Quadratic formula: \( x \) is the other root of the quadratic having \( x \) as root.

Thm: If \( x = \sqrt{a} + \sqrt{b} \), then
\[ x = \langle a_0, \ldots, a_n \rangle \text{ purely periodic} \]

Pf:
\[ x' = \sqrt{a} - \sqrt{b} \text{ so } -1 < x' < 0 \]

Set \( x = \langle a_0, \ldots, a_{k+1} \rangle = \langle a_0, \ldots, a_k, \frac{a_{k+1}}{\sqrt{a}} \rangle \) with
\[ a_{k+1} = \frac{1}{x} \text{ (so and so } a_k = L^x \text{), } x = \]

Then in your how you essentially show (since \( \sqrt{n} = \langle b_0, b_1, \ldots \rangle \) has \( b_0 = a_0 - \sqrt{b} \), \( b_k = a_k \) other \( k \))

\[ x_k = \frac{\sqrt{a} - M_k}{q_k} \text{ so } \xi_{k+1} = \frac{q_k}{\sqrt{a} - M_k} = \frac{\sqrt{a} + M_k}{q_{k+1}} \]

\[ x_{k+1} = \xi_{k+1} - q_{k+1} = \frac{\sqrt{a} - M_{k+1}}{q_{k+1}} \text{ where} \]
\[ q_{k+1} = n - M_k^2 \quad (\rightarrow \text{defines } q_{k+1}) \]
\[ a_{k+1} = L^\xi_{k+1} \]
\[ M_{k+1} = q_{k+1} q_{k+1} - a_k \quad (\rightarrow \text{defines } M_{k+1}) \]
Note that
\[ \xi_{k+1} = \frac{1}{\xi_k - a_k} = \frac{1}{\sqrt{\nu} + \frac{M_k - 1}{\gamma_k}} \]

Then
\[ \xi_k' = \frac{-\sqrt{\nu} + M_k - 1}{\gamma_k} \quad \text{and} \quad \frac{1}{\xi_k' - a_k} = \frac{q_k}{(M_k - 1 - a_k q_k) - \sqrt{\nu}} = \frac{q_k (\frac{(M_k - 1 - a_k q_k) - \sqrt{\nu}}{\gamma_k})}{(M_k - 1 - a_k q_k)^2 - \nu} = \xi_k' \]

But \( x = \xi_0 \Rightarrow \xi_k' - 1 < \xi_k' < 0 \), and then by induction \( -1 < \xi_k' < 0 \Rightarrow \xi_k' - a_k < -1 \Rightarrow 0 < \xi_k' < \xi_{k+1} < 0 \)

So \( x - 1 < \xi_k' < 0 \) for all \( k \), so
\[ \left| \xi_{k+1} \right| = \left| \frac{1}{\xi_k - \xi_k'} \right| = q_k \left| \right. \left. \text{since } q_k < q_k - \xi_k' < q_k + 1 \right. \]

But now, from HW, we know that the old fraction \( \frac{\xi_k}{\nu} \), and so for \( \frac{\xi_k + k \lambda}{\nu} \), becomes periodic; so for some \( n \geq 0 \), \( \alpha_{k+1} = \alpha_k + n \lambda \) for some \( k \geq 0 \). Let \( m, n \) be the smallest such values. Then claim \( n = 0 \). I.e. \( \alpha_k = \alpha_{k+n} \)
This is because if \( \xi_n = 3^m \nu_n \rightarrow \xi'_n = 3^m \nu'_n \)

\[ \mathbb{E} = \left\lfloor \frac{-1}{3^m} \right\rfloor = a_{n-1} = a_{n+m-1} = \left\lfloor \frac{-1}{\xi'_n} \right\rfloor \]

\[ \Rightarrow \frac{1}{\xi_{n-1} - a_{n+1}} = \xi_n = \xi_{n+m} \cdot \frac{1}{\xi_{n+m-1} - a_{n-1}} = \frac{1}{\xi_{n+m-1} - a_{n-1}} \]

\[ \Rightarrow \xi_n - a_{n+1} = \xi_{n+m-1} - a_{n-1} \Rightarrow \xi_{n+1} = \xi_{n+m-1} \]

contradicting choice of \( m \).

So if \( n = 0 \), \( a_n = a_s \) for all \( s \), \( s \geq n \)

\( \nu + L \alpha_s = \langle a_0, \ldots, a_{m-1} \rangle = \langle a_0, a_1, \ldots, a_{m-1}, a_0 \rangle \)

Note \( a_0 = 2L\alpha_L \), so

\( \nu = \langle L\alpha_L, a_1, \ldots, a_{m-1}, 2L\alpha_L \rangle \)

Note that: \( \xi_0 = \frac{\nu + L \alpha_L}{1} \Rightarrow m_0 = L \alpha_L \)

Thus since \( \xi_0 = \xi_n = \xi_{n+m} \Rightarrow \mathfrak{m} = 1 \) for all \( t \).