Math 445 Homework 9

Due Monday, December 9

38. (NZM, Problem 5.6.4) Let $f(x,y)$ be a polynomial with real coefficients and degree d, and set

$$p(t) = (1+t^2)^d f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)$$

[Note: $\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)$ is a popular parametrization of the unit circle].
Show that $p(t)$ has degree at most $2d$. Conclude that if $\mathcal{C}_f(\mathbb{R})$ meets the unit circle $\{(x,y) : x^2+y^2=1\}$ in more than $2d$ points, then it *contains* the unit circle.

39. (NZM, Problem 5.6.7) Show that if the curve

$$y^2 = ax^3 + bx^2 + cx + d = p(x)$$

(where $a,b,c,d \in \mathbb{R}$) has a double point, then it is of the form $(r,0)$, where r is a double root of $p(x)$.

40. Show that the curve

$$y^2 = x^3 - 4x^2 - 3x + 18$$

has a double point, and use the lines through this point to find the rational points on the curve.

41. (NZM, Problem 5.7.7) Let $\mathbf{A}$ and $\mathbf{B}$ be distinct points on the elliptic curve $\mathcal{C}_f(\mathbb{R})$, and suppose that the line through $\mathbf{A}$ and $\mathbf{B}$ is tangent to $\mathcal{C}_f(\mathbb{R})$ at $\mathbf{B}$. Show that

$$\mathbf{A} + 2\mathbf{B} = \mathbf{00}$$

42. (NZM, Problem 5.7.18) The cubic curve

$$axy = (x+1)(y+1)(x+y+b)$$

has three points at infinity; find them.