33. Three solutions to $x^2 + 3y^2 = 7z^2$.

Find rational solutions to $(\frac{x}{z})^2 + 3(\frac{y}{z})^2 = 7$, i.e. $x^2 + 3y^2 = 7$.

One solution (by inspection) is $x = 2, y = 1$. To find others, set $y = r(x-2) + 1$ with $r \in \mathbb{Q}$. Plugging in,

$$x^2 + 3(r(x-2)+1)^2 = 7 = x^2 + 3r^2(x-2)^2 + 6r(x-2) + 3$$

so

$$(x^2 - 4) + (x-2)(3r^2(x-2) + 6r) = 0$$

and

$$(x-2)((x+2) + 3r^2(x-2) + 6r) = 0$$

so $x = 2$ or $x = \frac{6r^2 + 6r - 2}{3r^2 + 1}$

Then

$$y = r(x-2) + 1 = r\left(\frac{6r^2 + 6r - 2}{3r^2 + 1} - 2\right) + 1$$

$$= r\left(\frac{6r^2 - 6r - 2 - 6r^2 - 2}{3r^2 + 1}\right) + 1 = \frac{-6r^2 - 4r + 3r^2 + 1}{3r^2 + 1}$$

$$= \frac{1 - 4r - 3r^2}{3r^2 + 1}$$

Setting $r = -\frac{a}{b}$ (we will want $ab > 0$ so that $xy,z > 0$!),

$$\frac{x}{z} = x = \frac{6a^2 + 6ab - 2b^2}{3a^2 + b^2}, \quad y = \frac{b^2 + 4ab - 3a^2}{3a^2 + b^2}$$

For example, with $a = b = 1$, we get

$$\frac{x}{z} = \frac{10}{4} = \frac{5}{2}, \quad \frac{y}{z} = \frac{2}{4} = \frac{1}{2}, \quad \text{so } \begin{cases} x = 5, y = 1, z = 2 \end{cases} \text{ works; }$$

$$25 + 3 = 28 \checkmark$$

With $a = 1, b = 2$ we get

$$\frac{x}{z} = \frac{10}{7}, \quad \frac{y}{z} = \frac{9}{7}, \quad \text{so } \begin{cases} x = 10, y = 9, z = 7 \end{cases} \text{ works; }$$

$$100 + 243 = 343 \checkmark$$
Our original solution is \(\sqrt{\frac{x=2, y=1, z=1}{4 + 3 = \sqrt{7}}}\).

If I get bored, I will write a bunch more solutions after the rest of the problems.

34. Show that \(4x^2 + 11y^3 = 29\) has no integer solutions.

First try working mod 4! \(11y^3 \equiv 3 \mod{4}\) \(4x^2 + 11y^3 \equiv 29 \equiv 1\)

Thus \(3.11y^3 \equiv 33y^3 \equiv y^3 \equiv 3\mod{4}\). But \(y=3\) works. Q.E.D.

Try working mod 11!

\[
4x^2 \equiv 4x^2 + 11y^3 \equiv 29 \equiv 7
\]

Then \(3.4x^2 \equiv 12x^2 \equiv x^2 \equiv 3.7 \equiv 21 \equiv 10 \equiv -1\).

But \(x^2 \equiv -1\) has no solutions; Euler's Criterion says to see if (since 11 is prime)

\[
(-1)^{\frac{11-1}{2}} = (-1)^5 = -1 \equiv 1
\]

For: just check:

\[
6^2 \equiv 0, 1^2 \equiv 1, 2^2 \equiv 4, 3^2 \equiv 9, 4^2 \equiv 16 \equiv 5, 5^2 \equiv 25 \equiv 3,
\]

\[
6^2 \equiv 36 \equiv 3, 7^2 \equiv 49 \equiv 5, 8^2 \equiv 64 \equiv 9, 9^2 \equiv 81 \equiv 4, 10^2 \equiv 100 \equiv 1
\]

Note that we really only need to compute \(x^2 \mod{11}\) after that the resulting values repeat themselves (in the opposite order).
35. Show that \( 57x^2 + 113y^2 = 116z^2 \) has no solution with \( xy7 \neq 0 \).

The only coefficient that is prime is 113.

(\( \frac{57}{13}, \frac{113}{2}, \frac{113}{3}, \frac{113}{7} \), so no prime \( \sqrt{113} \) is a factor.)

Consider the equation mod 113:

\[
\begin{align*}
57x^2 &\equiv 57x^2 + 113y^2 \\ &\equiv 116z^2 \\ &\equiv 3z^2.
\end{align*}
\]

Thus

\[
2 \cdot 57x^2 \equiv 114x^2 \equiv 23x^2 \equiv 6z^2.
\]

Note that \( 113 | 7 \implies 113 | x \), so \( 113y^2 = 116z^2 - 57x^2 \) is a multiple of \( (113)^2 \), so \( 113 | y^2 \) so \( 113 | y \), so \( (x, y, z) \) is not a primitive solution. But if the equation has a solution, then (dividing \( x, y, z \) by \( (x, y, z) \)) it has a primitive one. So we may assume that \( z \neq 0 \).

But then there is a \( \bar{z} \) with \( \bar{z} \bar{z} \equiv 1 \) (\( \bar{z} = 7^{-1} \) works!)

So

\[
\bar{z}^2 \equiv (\bar{z}x)^2 \equiv 6 \bar{z}^2 \bar{x}^2 \equiv 6 \bar{z} (\bar{z}^2)^2 \equiv 6 \bar{z} (1)^2 \equiv 6.
\]

So \( 6 \) is a square mod 113.

But it isn't! \( \bar{z}^2 \equiv 6 \) has a solution \( \iff \)

\[
6 \bar{z} \equiv 1.
\]

But

\[
56 = 32 + 24 = 32 + 16 + 8.
\]

and

\[
6 \equiv 6, \ 6^2 \equiv 36, \ 6^4 \equiv (36)^2 \equiv 1296 \equiv 11 \cdot 113 + 53 \equiv 53.
\]
\[ 6^8 \equiv 53^2 = 2809 = 113 \cdot 24 \frac{1}{2} \equiv 97 \equiv 97 \]
\[ 6^{16} \equiv 97^2 = 9409 = 113 \cdot 83 + 30 \equiv 30 \]
\[ 6^{22} \equiv 3^2 = 9 = 113 \cdot 113 - 4 \equiv -4 \]

So
\[ 6^{56} \equiv 97 \cdot 30 \cdot (-4) \equiv -(113)(97) \equiv -11640 \equiv -(113 \cdot 103 + 1) \]
\[ \equiv -1 \pmod{113} \]

So, 6 is not a square mod 113.

36. If \( x^2 + y^2 + z^2 = 2xyz \), then \( x = y = z = 0 \).

Note that if one of them is 0, then \( x^2 + y^2 + z^2 = 0 \) which implies all of them are 0. So suppose none of \( x, y, z \) are 0. Note that \( x^2 + y^2 + z^2 \) even \( \Rightarrow \) either 2 of them or two of \( x, y, z \) are odd; in particular, at least one is even, so \( x^2 + y^2 + z^2 \equiv 2xyz \equiv 0 \). But if, say, \( x \) is even and \( y, z \) are odd, then \( x^2 \equiv 0 \), \( y^2 \equiv z^2 \equiv 1 \), so \( x^2 + y^2 + z^2 \neq 0 \).

So it can't be the case that any of \( x, y, z \) are odd, i.e., they are all even. So set \( x = 2x_1, y = 2y_1, z = 2z_1 \), then we have
\[ 4x_1^2 + 4y_1^2 + 4z_1^2 = 2(8x_1y_1z_1), \text{ i.e. } \ x^2 + y^2 + z^2 = 4x_1y_1z_1. \]

But considering this equation mod 4 will again lead us to conclude that \( x = 2x_1, y = 2y_1, z = 2z_1 \), and then
\[ x^2 + y^2 + z^2 \equiv 8x_2y_2z_2 \text{ mod } 8. \]

This cannot continue! One way to make this precise is
to consider the equations $x^2 + y^2 + z^2 = 2^k x y z$, $k \geq 1$, together. If we choose the $k=1$ with the solution $x,y,z \geq 1$ with smallest $x$ (say), then our argument above implies that $x=2x_1, y=2y_1, z=2z_1$ with $x_1^2 + y_1^2 + z_1^2 = 2^{k-1} x_1 y_1 z_1$ and $x_1 < x$, a contradiction. So none of the equations can have a solution with $x,y,z \geq 1$.

$$x^2 + y^2 + z^2 = 2^k x y z \implies x = y = z = 0.$$

37. If $p$ prime, then $(x^2 - 17)(x^2 - 19)(x^2 - 323) \not\equiv 0 \pmod{p}$ has a solution.

If $p=2$, take $x=1$: $1^2 - 17 = -16 \equiv 0 \pmod{2}$, so the product will be $\equiv 0 \pmod{2}$.

If $p>2$, then $p$ is odd, and we know by Euler's criterion that $x^2 \equiv a \pmod{p}$ has a solution $\iff a^{(p-1)/2} \equiv 1 \pmod{p}$. But since $a^{(p-1)/2} \equiv 1 \pmod{p}$, we know that

$a^{(p-1)/2} \equiv 1 \pmod{p}$.

So if $x^2 \equiv 17 \pmod{p}$ has no solution and $x^2 \equiv 19 \pmod{p}$ has no solution, then $17^{(p-1)/2} \equiv -1 \pmod{p}$ and $19^{(p-1)/2} \equiv -1 \pmod{p}$.

But then $323^{(p-1)/2} = (17 \cdot 19)^{(p-1)/2} = 17^{(p-1)/2} \cdot 19^{(p-1)/2} \equiv (-1)(-1) = 1 \pmod{p}$, so $x^2 \equiv 323 \pmod{p}$ has a solution. So for every $p$, there is an $x$ so that either $p \mid x^2 - 17$ or $p \mid x^2 - 19$ or $p \mid x^2 - 323$, so there is an $x$ so that $p \mid (x^2 - 17)(x^2 - 19)(x^2 - 323)$, i.e.,

$$(x^2 - 17)(x^2 - 19)(x^2 - 323) \equiv 0 \pmod{p}$$

has a solution. \hfill \blacksquare
I wasn't really bored, but you folks came up with a lot of different solutions. Among them:

\[(5, 1, 2), (34, 3, 13), (82, 1, 31), (10, 9, 7), \]
\[(10, 47, 31), (10, 19, 13), (59, 47, 38), (17, 19, 19), \]
\[(17, 3, 12), (17, 87, 74), (86, 27, 37), (50, 3, 19)\]

Just for fun, I decided to try \(a = 47, b = 23\), which

gives \((9341, 887, 3578)\) (after dividing by the gcd).