Math 445 H.W. #6 Solutions

25. Show that there is an irrational number x so that

$$\left|x-\frac{a}{b}\right| < \left|x-\frac{m}{k_n}\right| \quad \text{and} \quad b<k_{n+1} \text{ for suitable values of } a,b.$$

$x=\sqrt{2}=\langle 1,\overline{2}\rangle$ works: its first few convergents are

$\frac{0}{1}, \frac{1}{0}, \frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots$ and, since $\sqrt{2}=1.415\ldots$

$\left|\sqrt{2}-\frac{3}{2}\right| = |1.5-1.415\ldots| > .084$, while

$\left|\sqrt{2}-\frac{4}{3}\right| = |1.415\ldots-1.333\ldots| < .083$, so $\left|\sqrt{2}-\frac{4}{3}\right| < \left|\sqrt{2}-\frac{3}{2}\right|$

and $3<5$.

<u>OR</u>

$x=\sqrt{3}=\langle 1,\overline{1,2}\rangle$ works: the first few convergents are

$\frac{0}{1}, \frac{1}{0}, \frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \ldots$ and since $\sqrt{3}=1.732\ldots$

$\left|\sqrt{3}-\frac{2}{1}\right| = |2-1.732\ldots| > .267$, while

$\left|\sqrt{3}-\frac{3}{2}\right| = |1.732-1.5| < .233$, so

$\left|\sqrt{3}-\frac{3}{2}\right| < \left|\sqrt{3}-\frac{2}{1}\right|$ and $2<3$.

It seems like nearly any (irrational) x would work?

26. Two solutions to $x^2 - 21y^2 = 1$

$4 < \sqrt{21} < 5$:

$a_0 = 4 \quad x_0 = \sqrt{21} - 4$

$\xi_1 = \frac{\sqrt{21}+4}{5} \quad a_1 = 1 \quad x_1 = \frac{\sqrt{21}-1}{5}$

$\xi_2 = \frac{\sqrt{21}+1}{4} \quad a_2 = 1 \quad x_2 = \frac{\sqrt{21}-3}{4}$

$\xi_3 = \frac{\sqrt{21}+3}{3} \quad a_3 = 2 \quad x_3 = \frac{\sqrt{21}-3}{3}$

$\xi_4 = \frac{\sqrt{21}+3}{4} \quad a_4 = 1 \quad x_4 = \frac{\sqrt{21}-1}{4}$

$\xi_5 = \frac{\sqrt{21}+1}{5} \quad a_5 = 1 \quad x_5 = \frac{\sqrt{21}-4}{5}$

$\xi_6 = \sqrt{21}+4 \quad a_6 = 8 \quad x_6 = \frac{\sqrt{21}-4}{1} = x_0$

$\Rightarrow \frac{3524}{769}, \frac{6049}{1320}$
$(-5), (1)$

Convergents: $\frac{0}{1}, \frac{1}{0}, \frac{4}{1}, \frac{5}{1}, \frac{9}{2}, \frac{23}{5}, \frac{32}{7}, \frac{55}{12}, \frac{472}{103}, \frac{527}{115}, \frac{999}{218}, \frac{2525}{551},$

value of $h_m^2 - 21k_m^2$: (1) (-5) (4) (-3) (4) (-5) (1) (5) (4) (-3) (4)

So $55^2 - 21 \cdot (12)^2 = 1$ $\quad$ $\boxed{x = 55, y = 12}$ is a solution

From the above (computing convergents), $\boxed{x = 6049, y = 1320}$ is also a solution.

OR: $(55 + \sqrt{21} \cdot 12)^2 = (55^2 + 21 \cdot 12^2) + \sqrt{21}(2 \cdot 55 \cdot 12)$

$= (3025 + 3024) + \sqrt{21}(110 \cdot 12)$

$= 6049 + \sqrt{21} \cdot 1320$

So $x = 6049, y = 1320$ is a solution.

27. For which $1 \le N \le \sqrt{33}$ does $x^2 - 33y^2 = N$ have a solution?

$5 < \sqrt{33} < 6$:

$$a_0 = 5 \quad x_0 = \sqrt{33} - 5$$

$$\xi_1 = \frac{\sqrt{33}+5}{8} \quad a_1 = 1 \quad x_1 = \frac{\sqrt{33}-3}{8}$$

$$\xi_2 = \frac{\sqrt{33}+3}{3} \quad a_2 = 2 \quad x_2 = \frac{\sqrt{33}-3}{3}$$

$$\xi_3 = \frac{\sqrt{33}+3}{8} \quad a_3 = 1 \quad x_3 = \frac{\sqrt{33}-5}{8}$$

$$\xi_4 = \sqrt{33}+5 \quad a_4 = 10 \quad x_4 = \sqrt{33}-5 = x_0$$

So $\sqrt{33} = \langle 5, \overline{1,2,1,10} \rangle$. The values of $h_m^2 - 33k_m^2$ will be:

$m=0$	-8
$m=1$	3
$m=2$	-8
$m=3$	1

and then <u>repeat</u>.

So among $1,2,3,4,5 \le \sqrt{33}$, 1 and 3 will occur as values of $x^2 - 33y^2$; 4 will occur because it is a perfect square ($2^2 - 33(0)^2 = 4$). Since 2 and 5 <u>cannot</u> occur as values of $h_m^2 - 33k_m^2$ for any m,

$x^2 - 33y^2 = N$ has <u>no</u> solutions for $N = 2, 5$

(also, for $N = -1, -2, -3, -4, -5$)

$x^2 - 33y^2 = N$ <u>has</u> solutions for $N = 1, 3, 4$.