25. Show that there is an irrational number $x$ so that 

$$|x - \frac{a}{b}| < |x - \frac{m}{k_n}|$$

and $b < k_n$, for suitable values of $a, b$.

$x = \sqrt{2} = \langle 1, 1, 2 \rangle$ works: its first few convergents are 

$$\varphi, \frac{1}{1}, \frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{12}{9}, \ldots$$

and since $\sqrt{2} = 1.415\ldots$

$$|\sqrt{2} - \frac{3}{2}| = \left| 1.415 - 1.415 \right| > .084,$$ 

while 

$$|\sqrt{2} - \frac{11}{8}| = \left| 1.415 - 1.383 \right| < .033,$$ 

so $$|\sqrt{2} - \frac{3}{2}| > |\sqrt{2} - \frac{11}{8}|$$

and $3 < 5$.

$x = \sqrt{3} = \langle 1, 1, 2 \rangle$ works: the first few convergents are 

$$\varphi, \frac{1}{1}, \frac{1}{1}, \frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \ldots$$

and since $\sqrt{3} = 1.732\ldots$

$$|\sqrt{3} - \frac{2}{1}| = \left| 1.732 - 1.732 \right| > .267,$$ 

while 

$$|\sqrt{3} - \frac{3}{2}| = \left| 1.732 - 1.732 \right| < .233,$$ 

so 

$$|\sqrt{3} - \frac{2}{1}| < |\sqrt{3} - \frac{3}{2}|$$

and $2 < 3$.

It seems like nearly any (irrational) $x$ would work?
26. Two solutions to \( x^2 - 21y^2 = 1 \)

\[ y < \sqrt{21} : \]

\( a_0 = 1 \quad x_0 = \sqrt{21} - 1 \)

\[ \xi_1 = \frac{\sqrt{21} + 1}{4} \quad a_1 = 1 \quad x_1 = \frac{\sqrt{21} - 1}{2} \]

\[ \xi_2 = \frac{\sqrt{21} + 3}{5} \quad a_2 = 1 \quad x_2 = \frac{\sqrt{21} - 3}{4} \]

\[ \xi_3 = \frac{\sqrt{21} + 1}{4} \quad a_3 = 2 \quad x_3 = \frac{\sqrt{21} - 3}{3} \]

\[ \xi_4 = \frac{\sqrt{21} + 3}{5} \quad a_4 = 1 \quad x_4 = \frac{\sqrt{21} - 1}{4} \]

\[ \xi_5 = \frac{\sqrt{21} + 1}{4} \quad a_5 = 1 \quad x_5 = \frac{\sqrt{21} - 1}{5} \]

\[ \xi_6 = \sqrt{21} + 1 \quad a_6 = 8 \quad x_6 = \frac{\sqrt{21} - 1}{1} = x_0 \]

Convergents:

\( \frac{0}{1}, \frac{1}{0}, \frac{4}{1}, \frac{5}{1}, \frac{23}{2}, \frac{32}{5}, \frac{55}{12}, \frac{472}{103}, \frac{527}{115}, \frac{999}{218}, \frac{2551}{551}, \)

value of \( h_m^2 - 21k_m^2 \):

\( (1) (5) (4) (3) (4) (5) (4) (3) (4) \)

\( h = 55^2 - 21 \cdot 12^2 = 1 \quad \xi \sqrt{21}, y = 12 \) is a solution

From the above (computing convergents), \( x = 6049, y = 1320 \) is also a solution.

\( \xi \): \( (55 + \sqrt{21} \cdot 12)^2 = (55^2 + 21 \cdot 12^2) + 2(55 \cdot 12) \)

\( = (3025 + 3024) + 110 \cdot 12 \)

\( = 6049 + \sqrt{21} \cdot 1320 \)

\( x = 6049, y = 1320 \) is a solution.
27. For which $1 \leq N \leq 63$ does $x^2 - 33y^2 = N$ have a solution?

$5 < 6$

$x_0 = 5 
\frac{\sqrt{63} + 5}{8} 
\frac{\sqrt{63} + 3}{3} 
\frac{\sqrt{63} + 5}{8}$

$a_1 = 1 
\frac{\sqrt{63} - 3}{8} 
\frac{\sqrt{63} - 3}{3}$

$a_2 = 2 
\frac{\sqrt{63} - 5}{8}$

$a_3 = 1 
\frac{\sqrt{63} - 5}{8}$

$a_4 = 10 
\frac{\sqrt{63} - 5}{8} = x_0$

$s = \langle 5, 1, 2, 1, 10 \rangle$. The values of $h_m - 33k_m$ will be:

$m = 0 \quad -8$
$m = 1 \quad 3$
$m = 2 \quad -8$
$m = 3 \quad 1$ and then repeat.

So among $1, 3, 4, 5 \leq \sqrt{63}$, 1 and 3 will occur as values of $x^2 - 33y^2$; 4 will occur because it is a perfect square $(2^2 - 33(0)^2 = 4)$. Since 2 and 5 cannot occur as values of $h_m - 33k_m$ for any $m$,

$x^2 - 33y^2 = N$ has no solutions for $N = 2, 5$

(also, for $N = -3, -2, -3, -4, -5$)

$x^2 - 33y^2 = N$ has solutions for $N = 1, 3, 4$.