10. If \( p \mid q \) and \( a \equiv 1 \mod n \), then \( a^{p-1} \equiv 1 \mod n \).

For any \( n \geq 1 \),
\[
x^{n+1} - 1 = (x-1)(x^n + x^{n-1} + \cdots + x + 1) = (x-1) \sum_{k=0}^{n} x^k.
\]
(This can be proved by induction.
\[
x^{n+1} - 1 = (x^{n-1}x + x-1) = (x-1) \left( x \sum_{k=0}^{n-1} x^k + 1 \right) = (x-1) \sum_{k=0}^{n} x^k.
\]

If \( p \mid q \), then \( q = pn \) for some \( n \geq 1 \), so
\[
a^{q-1} = a^{pn-1} = (a^p)^n - 1 = \left( a^{p-1} \sum_{k=0}^{n-1} (a^p)^k \right) \mod n \text{ is a multiple of } a^p - 1, \text{ so } a^{p-1} \mid a^{p-1}.
\]

11. \((m,n) = 1\) if \( a^m \equiv 1 \mod m \), \( a^n \equiv 1 \mod n \), then \( a \equiv 1 \mod mn \).

Since \((p,q)\mid p\) and \((p,q)\mid q\), \((p,q)\mid p\) and \((p,q)\mid q\) are integers. Then
\[
a^{\frac{pq}{pq}} = (a^p)^{\frac{q}{pq}} \equiv \left( a^p \right)^q \equiv \left( a^p \right)^{\frac{q}{pq}} \equiv 1, \text{ and so } \left\lfloor \frac{pq}{pq} \right\rfloor \equiv a^{\frac{pq}{pq}} - 1, \text{ and}
\]
\[
a^{\frac{pq}{pq}} = (a^q)^{\frac{p}{pq}} \equiv \left( a^q \right)^p \equiv \left( a^q \right)^{\frac{p}{pq}} \equiv 1, \text{ so } \left\lfloor \frac{pq}{pq} \right\rfloor a^{\frac{pq}{pq}} - 1. \text{ But since}
\]
\((m,n) = 1\), this implies that \( mn \mid a^{\frac{pq}{pq}} - 1 \), i.e. \( a^{\frac{pq}{pq}} \equiv 1 \).

[Note just note that \( p \mid q \), so \( a^p \mid a^{\frac{pq}{pq}} - 1 \), to get the two cases....]

12. If \((m,n) = 1\) (and \((10,m) = (10,n) = 1\)) then
\[
\text{ord}_{mn}(10) = \frac{\text{ord}_m(10) \cdot \text{ord}_n(10)}{\text{gcd}(\text{ord}_m(10), \text{ord}_n(10))}
\]

For ease of notation, set \( r = \text{ord}_m(10) \), \( s = \text{ord}_n(10) \). Then
\( r \) is smallest positive integer with \( 10^r \equiv 1 \mod m \), and some for \( s \)
\( 10^s \equiv 1 \mod n \). By problem 11, we then know that \( 10^{\frac{rs}{\text{gcd}(r,s)}} \equiv 1 \mod mn \). Since
\[(m, n) = 1 \implies \text{ord}_{m} (10) \mid \frac{\text{rs}}{(r, s)} \]. To show that \(\text{ord}_{m} (10) = \frac{\text{rs}}{(r, s)}\), we then need to show that \(\frac{\text{rs}}{(r, s)} \mid \text{ord}_{m} (10)\), i.e., if \(10^k \equiv 1 \mod{m}\), then \(\frac{\text{rs}}{(r, s)} \mid k\). But \(10^{k} \equiv 1 \mod{m} \iff m \mid 10^{k-1} \iff (\text{since } (m, n) = 1) \mod{n} \mid 10^{k-1} \text{ and } n \mid 10^{k-1}. \) \(\implies (\equiv)\) is immediate, since \(m \mid mn \); \(\equiv\) uses \((m, n) = 1\). \(\implies (\equiv)\) is immediate, since \(m \mid mn \); \(\equiv\) uses \((m, n) = 1\). \(\implies (\equiv)\) is immediate, since \(m \mid mn \); \(\equiv\) uses \((m, n) = 1\). \(\implies (\equiv)\) is immediate, since \(m \mid mn \); \(\equiv\) uses \((m, n) = 1\). 

But \(m \mid 10^{k-1} \iff r = \text{ord}_{m} (10) \mid k\), and \(n \mid 10^{k-1} \iff s = \text{ord}_{m} (10) \mid k\). So \(10^{k} \equiv 1 \mod{m} \iff r \mid k\) and \(s \mid k\). But \(r \mid k\) and \(s \mid k\) \(\implies \frac{\text{rs}}{(r, s)} \mid k\), set \(k = ru\), \(k = sv\), then writing \((r, s) = rx + sy\) we have \((r, s)u = rxu + suy = (ru)x + sy\) = \(sv)x + suy = s(vx + uy)\), so \(u = \frac{r}{(r, s)}(vx + uy)\), so \(\frac{\text{rs}}{(r, s)} \mid k\). \(\implies (\equiv)\) is immediate, since \(m \mid mn \); \(\equiv\) uses \((m, n) = 1\). 

13. \((3, n) = (10, n) = 1 \implies \text{ord}_{3n} (10) = \text{ord}_{n} (10)\).

Since \((3, 1) = 1\) and \(\text{ord}_{3} (10) = 1\), \(\implies 10^l = 10 \equiv 1 \mod{3}\). But above we have
\[
\text{ord}_{3n} (10) = \frac{\text{ord}_{3} (10) \cdot \text{ord}_{n} (10)}{(\text{ord}_{3} (10), \text{ord}_{n} (10))} = \frac{1 \cdot \text{ord}_{n} (10)}{(1, \text{ord}_{n} (10))} = \frac{1}{1} = \text{ord}_{n} (10). \]
If we want to multiply \( b \) by \( g \), we must have \( \gcd(b, g) = 1 \).

So let's do it.

\[
\begin{align*}
10 \equiv 1 \pmod{7} \\
2 \equiv 2 \pmod{7} \\
\equiv 2 \pmod{7} \\
\equiv 2 \pmod{7} \\
\end{align*}
\]

So we choose 7.

\[
\begin{align*}
\phi(10) &= 4 \\
\phi(7) &= 6 \\
\end{align*}
\]

So we use 4.

We also know that

\[
\phi(7) = 6 - 7 \cdot \left( \frac{7}{7} \right) = 6
\]

To compute each \( (10)^{1001} \pmod{7} \) we have (10, 7) = 1, so

\[
\begin{align*}
1 &= 5 \cdot 10 + (-1) \cdot 7 \\
\end{align*}
\]

Since \( 10 \equiv 3 \pmod{7} \) and \( 5 \equiv 5 \pmod{7} \),

\[
\begin{align*}
10^3 &\equiv 3^3 \\
&\equiv 27 \\
&\equiv 6 \\
&\equiv 6 \pmod{7}
\end{align*}
\]

and hence

\[
\begin{align*}
10^4 &\equiv 6 \\
&\equiv 6 \pmod{7}
\end{align*}
\]

\[
\begin{align*}
10^5 &\equiv 5 \cdot 6 \\
&\equiv 30 \\
&\equiv 2 \pmod{7}
\end{align*}
\]

Therefore, for any \( a \) and \( b \pmod{10} \),

\[
\text{Compute } a^{1001} \pmod{7}. 
\]