1. Compute \((1819, 3587)\):

\[
\begin{align*}
3587 &= 1 \cdot 1819 + 1768 \\
1819 &= 1 \cdot 1768 + 51 \\
1768 &= 34 \cdot 51 + 34 \\
51 &= 1 \cdot 34 + 17 \\
34 &= 2 \cdot 17 + 0
\end{align*}
\]

So \((1819, 3587) = (1768, 1819) = \ldots = (17, 0) = 17\), and

\[
17 = 51 - 1 \cdot 34 = 51 - 1 \cdot (1768 - 34 \cdot 51) = 35 \cdot 51 - 1 \cdot 1768 = 35 \cdot 1819 - 36 \cdot 1768 = 35 \cdot 1819 - 36 \cdot (3587 - 1 \cdot 1819) = 71 \cdot 1819 + (-36) \cdot 3587
\]

[Check: \(71 \cdot 1819 - 36 \cdot 3587 = 129149 - 129132 = 17\) \(\checkmark\)]

\[
1819 = 17 \cdot 107, \quad 3587 = 17 \cdot 211
\]

2. For any \(n\), \(6 \mid n(n+1)(n+2)\) and \(24 \mid n(n+1)(n+2)(n+3)\).

For any \(n\), \(\frac{n}{6} \in \{0, 1, 3, 4, 5\}\). If

- \(n \equiv 0 \mod 6\), then \(6 \mid n\) so \(6 \mid n(n+1)(n+2) = P_3(n)\)
- \(n \equiv 1 \mod 6\), then \(n+1 \equiv 2 \mod 3\), so \(2 \mid n+1\), and \(n+2 \equiv 3 \mod 3\), so \(3 \mid n+2\), so \(2 \cdot 3 \mid P_3(n)\)
- \(n \equiv 3 \mod 6\), then \(n+1 \equiv 2 \mod 3\), so \(2 \mid n+1\) and \(3 \mid n+2\), so \(6 \mid P_3(n)\)
- \(n \equiv 4 \mod 6\), then \(n+1 \equiv 3 \mod 3\), so \(3 \mid n+1\) and \(2 \mid n+2\), so \(6 \mid P_3(n)\), as above.
- \(n \equiv 5 \mod 6\), then \(n+1 \equiv 4 \mod 3\), so \(3 \mid n+1\) and \(2 \mid n+2\), so \(6 \mid P_3(n)\), as above.
\( n \equiv 4 \), then \( 6 \mid n+2 \leq 0 \), so \( 6 \mid P_3(n) \).

\( n \equiv 5 \), then \( 6 \mid n+1 \leq 0 \), so \( 6 \mid P_3(n) \).

So no matter what value \( n \) has mod 6, \( 6 \mid P_3(n) = n(n+1)(n+2) \).

\[ P_4(n) = n(n+1)(n+2)(n+3) = P_3(n) \cdot (n+3) \]

Since \( 6 \mid P_3(n) \), \( 3 \mid P_3(n) \), so \( 3 \mid P_4(n) \).

By looking at \( n \) mod 8 (or better, mod 4), we can conclude that one of \( n, n+1, n+2, n+3 \) is always \( \equiv 2 \), while another is \( \equiv 0 \) mod 2 or of them and \( 4 \mid \) another, so \( 8 \mid P_4(n) \) for every \( n \).

Since \( 3 \mid P_4(n) \) and \( 8 \mid P_4(n) \) and \( (3,8)=1 \), \( 3 \cdot 8 = 24 \mid P_4(n) \).

General conjecture! For any \( n,k \), \( k! \mid n(n+1) \cdots (n+k-1) \) [which is true!]

3. If \( p \) is prime and \( p \equiv 1 \pmod{3} \), then \( p \equiv 1 \).

If \( 2 \mid p \equiv 0 \pmod{3} \) then \( p = 6x + a \) some integer \( x \).

So \( p = 3(2x) + a \).

\( p \equiv 1 \pmod{3} \) so \( p = 3x+1 \) some integer \( x \).

\( x \) is either even or odd; if \( x \) is odd \( x = 2n+1 \), then \( p = 3(2n+1) + 1 = 6n+4 = 2(3n+2) \) is even (mod \( \equiv 4 \)) so cannot be prime. Therefore, \( x \) must be even \( x = 2m \), so \( p = 3(2m)+1 = 6m+1 \), so \( p \equiv 1 \pmod{4} \).
9. If \( x, y, \alpha \in \mathbb{R} \) and \( x^2 + y^2 = \alpha^2 \), then \( 3|x \) or \( 3|y \).

Suppose not. Suppose \( 3 \nmid x \) and \( 3 \nmid y \), then:

\[
x = 3x' + a, \quad a = 1 \text{ or } 2 \quad \text{and} \quad y = 3y' + b, \quad b = 1 \text{ or } 2
\]

But then,

\[
x^2 = (3x' + a)^2 = 9(x')^2 + 6x'a + a^2 = 3(3(x')^2 + 2x'a) + a^2 \equiv a^2 \equiv \frac{a^2}{3} \quad \text{since} \quad 1^2 \equiv 1 \quad \text{and} \quad 2^2 \equiv 4 \equiv 1.
\]

Similarly, \( y^2 \equiv 1 \). So \( x^2 + y^2 \equiv 1 + 1 = 2 \). But \( \frac{2}{3} \equiv 2 \).

But that is impossible, since \( \frac{2}{3} \equiv 0, 1, \text{ or } 2 \Rightarrow \frac{2}{3} \equiv 0^2 = 1 \text{ or } 1^2 = 1 \text{ or } 2^2 = 4 \equiv 1 \), so \( \frac{2}{3} \equiv 0 \text{ or } 1 \).

So our assumption is false; so we must have either \( 3|x \) or \( 3|y \).
The more general conjecture: \( \forall n, \forall a \ n! \mid a(a+1) \cdots (a+n-1) \)

Proof by induction:

- \( n=1 \): \( 1! \mid a \) for all \( a \). \( \checkmark \) \( 1! = 1 \).
- Suppose true for \( n-1 \), show true for \( n \).
- Show \( n! \mid a(a+1) \cdots (a+n-1) \) \( \forall a \geq 1 \). By induction
  
  - \( a=1 \): \( n! \mid 1 \cdot 2 \cdots (1+n-1) = n! \). \( \checkmark \)

Suppose true for \( n \), show true for \( n+1 \).

We know \( n! \mid a(a+1) \cdots (a+n-1) \)

We want \( n! \mid (a+1) \cdots (a+n-1)(a+n) \)

But

\[
(a+1) \cdots (a+n) = (a+1) \cdots (a+n-1)a + (a+1) \cdots (a+n-1)n
\]

\[
= a(a+1) \cdots (a+n-1) + (a+1) \cdots ((a+1)+(n-1)-1)n
\]

By inductive hypothesis, \( n! \mid a(a+1) \cdots (a+n-1) \)

By other inductive hypothesis, \( (n-1)! \mid (a+1) \cdots ((a+1)+(n-1)-1) \)

So \( n! \mid n(a+1) \cdots ((a+1)+(n-1)-1) \)

So \( n! \) | their sum, i.e. \( n! \mid (a+1) \cdots ((a+1)+n-1) \)

So by induction \( n! \mid a \cdots (a+n-1) \) for all \( a \geq 1 \).

So by induction, \( \forall a \geq 1 \), \( n! \mid a \cdots (a+n-1) \) for all \( a \geq 1 \).