Math 445 Exam #2 Solutions

1. $x \notin \mathbb{Q}$ h_n/k_n = convergent, then $k_n|xk_{n-1}-h_{n-1}|+k_{n-1}|xk_n-h_n|=1$.

Divide by k_nk_{n-1}, it is then enough to show that

$$\left|x-\frac{h_{n-1}}{k_{n-1}}\right|+\left|x-\frac{h_n}{k_n}\right|=\frac{1}{k_nk_{n-1}}.$$ But we know

that for any n, either $\frac{h_{n-1}}{k_{n-1}}<x<\frac{h_n}{k_n}$ or $\frac{h_n}{k_n}<x<\frac{h_{n-1}}{k_{n-1}}$

(the convergents alternate which side of x they are on).

So $x-\frac{h_{n-1}}{k_{n-1}}$ and $x-\frac{h_n}{k_n}$ have opposite sign. So

$x-\frac{h_{n-1}}{k_{n-1}}$ and $\frac{h_n}{k_n}-x$ have the same sign, and

$$\left|x-\frac{h_{n-1}}{k_{n-1}}\right|+\left|x-\frac{h_n}{k_n}\right|=\left|x-\frac{h_{n-1}}{k_{n-1}}\right|+\left|\frac{h_n}{k_n}-x\right|$$

$$=\left|x-\frac{h_{n-1}}{k_{n-1}}+\frac{h_n}{k_n}-x\right|=\left|\frac{h_n}{k_n}-\frac{h_{n-1}}{k_{n-1}}\right|=\left|\frac{h_nk_{n-1}-h_{n-1}k_n}{k_nk_{n-1}}\right|$$

$$=\left|\frac{(-1)^n}{k_nk_{n-1}}\right|=\frac{1}{k_nk_{n-1}},$$ as desired.

7. Continued fraction of $\sqrt{39}$:

$$\lfloor\sqrt{39}\rfloor = 6 \quad \sqrt{39} = 6 + (\sqrt{39}-6) = a_0 + x_0$$

$$\frac{1}{\sqrt{39}-6} = \frac{\sqrt{39}+6}{3} \quad \left\lfloor\frac{\sqrt{39}+6}{3}\right\rfloor = 4 \quad \frac{\sqrt{39}+6}{3} = 4 + \left(\frac{\sqrt{39}-6}{3}\right) = a_1 + x_1$$

$$\frac{3}{\sqrt{39}-6} = \sqrt{39}+6 \quad \lfloor\sqrt{39}+6\rfloor = 12 \quad \sqrt{39}+6 = 12 + (\sqrt{39}-6) = a_2 + x_2$$

$x_2 = x_0$, so everything will repeat; $\sqrt{39} = \langle 6, \overline{4, 12}\rangle$.

Solve $x^2 - 39y^2 = 1$ $\quad (x,y) = (h_n, k_n)$ for some n

Denom of x_2 is 1, so $h_1^2 - 39k_1^2 = 1\cdot(-1)^{1+1} = 1$; same for x_4.

$h_{-2} = 0,\ h_{-1} = 1,\ h_0 = 6,\ h_1 = 25,\ h_2 = 306,\ h_3 = 1249$

$k_{-2} = 1,\ k_{-1} = 0,\ k_0 = 1,\ k_1 = 4,\ k_2 = 49,\ k_3 = 200$

So $(25,4)$ is a solution, so is $\underline{(1249, 200)}$.

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or $(25,4)$ is a solution, so computing

$$\begin{aligned}(25+4\sqrt{39})^2 &= ((25)^2 + 39\cdot 16) + (2\cdot 4\cdot 25)\sqrt{39} \\ &= (625 + 624) + 200\sqrt{39} \\ &= 1249 + 200\sqrt{39}\end{aligned}$$

implies that $(1249, 200)$ is a solution.

Among $N = 1,2,3,4,5$, only 1 and 3 occur as a denominator of x_n. But $h_{2n}^2 - 39k_{2n}^2 = -3$, not 3. So only 1 (from convergents) and 4 (as a perfect square) will ~~give~~ have solutions.

3. $x^2 - 39y^2 = 1776$ has no integer solutions.

$39 = 3 \cdot 13$, try modulo $\underline{\underline{3}}$:

$$x^2 - 39y^2 \underset{3}{\equiv} x^2 \underset{3}{\equiv} 1776 \equiv 0 \quad \underline{\text{has}} \text{ solutions } (3 | x).$$

Try modulo $\underline{\underline{13}}$:

$$x^2 - 39y^2 \underset{13}{\equiv} x^2 \underset{13}{\equiv} 1776 = 13 \cdot 136 + 8 \underset{13}{\equiv} 8$$

Does $x^2 \underset{13}{\equiv} 8$ have solutions? 13 is an odd prime, so using Euler's Criterion,

$\underline{\text{Need}}$ $8^{\frac{13-1}{2}} = 8^6 \underset{13}{\equiv} 1$. But $8 = 2^3$ so

$$8^6 \underset{13}{\equiv} 2^{18} \underset{13}{\equiv} 2^{12} \cdot 2^6 \underset{13}{\equiv} 1 \cdot 2^6 \underset{13}{\equiv} 8^2 \underset{13}{\equiv} 64 \underset{13}{\equiv} -1.$$

So $x^2 - 39y^2 \underset{13}{\equiv} 1776$ has no solutions, so

$$x^2 - 39y^2 = 1776 \text{ has no solutions.}$$

4. Find the solutions to $x^2+3y^2=19$ with $x,y\in\mathbb{Q}$.

By inspection, $x_0=4, y_0=1$ is a solution. To find all others (since a line through (x_0,y_0) and (x,y) would have rational slope) set $y=r(x-x_0)+y_0 = r(x-4)+1$ for $r\in\mathbb{Q}$. Then plug in:

$$x^2+3(r(x-4)+1)^2 = 19 = x^2+3r^2(x-4)^2+6r(x-4)+3$$

$$(x^2-16)+(x-4)(3r^2(x-4)+6r)=0=(x-4)((x+4)+3r^2(x-4)+6r)$$

$$=(x-4)(x(3r^2+1)-(12r^2-6r-4)), \text{ so}$$

$x=4$ or $x=\dfrac{12r^2-6r-4}{3r^2+1}$. Then

$$y=r\left(\frac{12r^2-6r-4}{3r^2+1}-4\right)+1 = r\left(\frac{12r^2-6r-4-12r^2-4}{3r^2+1}\right)+1$$

$$=\frac{-6r^2-8r}{3r^2+1}+1=\frac{-6r^2-8r+3r^2+1}{3r^2+1}=\frac{-3r^2-8r+1}{3r^2+1}$$

So $(4,1)$, and $\left(\dfrac{12r^2-6r-4}{3r^2+1}, \dfrac{-3r^2-8r+1}{3r^2+1}\right)$ for $r\in\mathbb{Q}$ (and for $r=\infty$ (i.e., $r\to\infty$, $(4,-1)$)) are the rational solutions to $x^2+3y^2=19$. ///

5 $n \underset{8}{\equiv} 7$, then $n = x^2+y^2+z^2$ has no solutions.

Mod 8, $x^2 \underset{8}{\equiv} 0,1,4,1,0,1,4,1$, i.e. $x^2 \underset{8}{\equiv} 0, 1,$ or 4

So $x^2+y^2 \underset{8}{\equiv} \overbrace{0,1,4}^{0+\_}, \overbrace{1,2,5}^{1+\_}, \overbrace{4,5,0}^{4+\_}$, i.e. $x^2+y^2 \underset{8}{\equiv} 0,1,2,4,$ or 5

So $x^2+y^2+z^2 \underset{8}{\equiv} \cancel{0,1,2,4,5,1,2,3,5,6,2,3},$

$$\underbrace{0,1,2,4,5}_{\_+0},\ \underbrace{1,2,3,5,6}_{\_+1},\ \underbrace{4,5,6,0,1}_{\_+4}$$

i.e. $x^2+y^2+z^2 \underset{8}{\equiv} 0,1,2,3,4,5,$ or 6. So

$x^2+y^2+z^2$ is never $\underset{8}{\equiv}$ to 7, so if $n \underset{8}{\equiv} 7$, then

$x^2+y^2+z^2 = n$ is impossible.

This means, for example, that (since $15 \underset{8}{\equiv} 7$)

$15 = 3 \cdot 5$ cannot be expressed as the sum of 3 squares.

But $3 = 1^2+1^2+1^2$, and $5 = 2^2+1^2+0^2$, 15 <u>is</u> the product of two sums of 3 squares. So the product of two sums of 3 squares <u>cannot</u> always be expressed as a sum of 3 squares. //

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6. For $n, m \in \mathbb{Z}$ if $x^2+2y^2=m$ and $u^2+2v^2=n$ have solutions, then $z^2+2w^2=mn$ has a solution.

We will <u>build</u> (z,w) out of (x,y) and (u,v). If $x^2+2y^2=m$, and $u^2+2v^2=n$, then

$$mn=(x^2+2y^2)(u^2+2v^2)=x^2u^2+2y^2u^2+2x^2v^2+4y^2v^2$$

$$=(xu)^2+(2yv)^2+2((yu)^2+(xv)^2)$$

$$=(xu)^2+(2yv)^2+2((yu)^2+(xv)^2)+4xuyv-4xuyv$$

$$=(xu)^2+2((xu)(2yv))+(2yv)^2+2((yu)^2-2(yu)(xv)+(xv)^2)$$

$$=(xu+2yv)^2+2(yu-xv)^2$$

So if we set $z=xu+2yv$, $w=yu-xv$, then

$$z^2+2w^2=mn.$$