Math 417 Problem Set 9

Starred (*) problems are due Friday, April 15.

68. Show that if $\varphi : G \to H$ is a homomorphism, and $N \leq H$ is a normal subgroup of H, then $\varphi^{-1}(N) = \{x \in G \ : \ \varphi(x) \in N\}$ is a normal subgroup of G, and $G/\varphi^{-1}(N) \cong \varphi(G)/[\varphi(G) \cap N]$.

(*) 69. Show that in the symmetric group S_n, every commutator $\alpha\beta\alpha^{-1}\beta^{-1}$ is an element of the subgroup A_n = the alternating group. Show, in addition, that every 3-cycle (a, b, c) can be written as a commutator $\alpha\beta\alpha^{-1}\beta^{-1}$. Conclude that every element of A_n can be written as a product of commutators.

70. Use problem #68 to show that if $\varphi : S_n \to G$ is a homomorphism from the symmetric group to an abelian group G, then $\varphi(A_n) = \{e_G\}$ and so $\varphi(S_n)$ is either the trivial subgroup or isomorphic to $\mathbb{Z}_2$.

71. (Gallian, p.415, # 5 (sort of)) If $|G| = 36 = 2^2 \cdot 3^2$ and G has a 2-Sylow subgroup H and a 3-Sylow subgroup K that are both normal, show that the "natural" homomorphism $G \to G/H \oplus G/K$ given by $x \mapsto (xH, xK)$ is an isomorphism, and conclude (from earlier results) that G must be abelian.

(*) 72. (Gallian, p.416, # 33) If $|G| = p^n$ with p prime, show that for every k, $1 \leq k \leq n$, there is a normal subgroup $N \leq G$ with $|N| = p^k$.

[Hint: take the quotient by some element of the center of G, and use induction!]

73. Sylow's first theorem implies that the symmetric group S_5 (with $|S_5| = 5! = 2^3 \cdot 3 \cdot 5$) has a subgroup of order 8. Find one! Is the subgroup you found a normal subgroup of S_5?

(*) 74. In class we (essentially) showed that for p a prime, $|GL(2, \mathbb{Z}_p)| = p(p-1)(p^2-1)$. So, for example, $|GL(2, \mathbb{Z}_5)| = 5 \cdot 4 \cdot 24 = 480$, and so $GL(2, \mathbb{Z}_5)$ must have elements of order 3 and of order 5. Find some! Are the subgroups that they generate normal?

75. (Gallian, p.432, # 26) Let G be a finite group that is simple, that is, the only normal subgroups of G are $\{e_G\}$ and G. Suppose that $H, K \leq G$ are subgroups of G and $[G : H] = p$ and $[G : K] = q$ are both prime. Show that $p = q$.

[Hint: if not, then WOLOG $p < q$; then show that the action of G on G/H enables you to find a normal subgroup of G with index relatively prime to q.]