Math 417 Problem Set 8

Starred (*) problems are due Friday, April 8.

60. If G is a group, show that $H = \{(g, g) \ : \ g \in G\}$ is a normal subgroup of $G \oplus G \Leftrightarrow G$ is abelian; when H is normal, show that $(G \oplus G)/H$ is isomorphic to G.

    [Hint: how would you build a homomorphism $G \oplus G \to G$ so that H would be the kernel?]

(*) 61. (Gallian, p.202, # 37) If H is a normal subgroup in G and G is finite, and $g \in H$, show that the order of gH in G/H divides the order of g in G.

62. If $\varphi : G \to H$ is a surjective homomorphism and $N \leq G$ is a normal subgroup of G, show that $\varphi(N) \leq H$ is a normal subgroup of H. Show, on the other hand, that if φ is not surjective, then $\varphi(N)$ need not be a normal subgroup.

63. (Gallian, p.238, # 5) Show that for any group G, the group of inner automorphisms $\mathrm{Inn}(G)$ of G is a normal subgroup of the group $\mathrm{Aut}(G)$ of automorphisms of G.

(*) 64. (Gallian, p.239, # 15) Show that if H and K are abelian, normal subgroups of the group G, and $H \cap K = \{e_G\}$, then the subgroup $N = HK$ is also abelian.

    [Hint: if $a, b \in HK$, show that $aba^{-1}b^{-1} \in H \cap K$.]

(*) 65. Show that 2 is not a generator for the group $\mathbb{Z}_{31}^*$ of units modulo 31, but that 3 is. If, using $\mathbb{Z}_{31}^*$ and $a = 3$ as the basis for a (very weak!) Diffie-Hellman key exchange, if Alice chooses $n = 5$ and Bob chooses $m = 11$ to build a shared key, what information do they send to one another and what is that key?

66. In the group S_{10} the elements $a = (1, 2, 3)(4, 5)(8, 9)$ and $b = (2, 4, 8)(1, 10)(3, 7)$ are conjugate. Find at least two distinct conjugating elements x (so that $xa = bx$).

67. Find a matrix $X \in GL(2, \mathbb{Z})$ so that $X \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} X$.